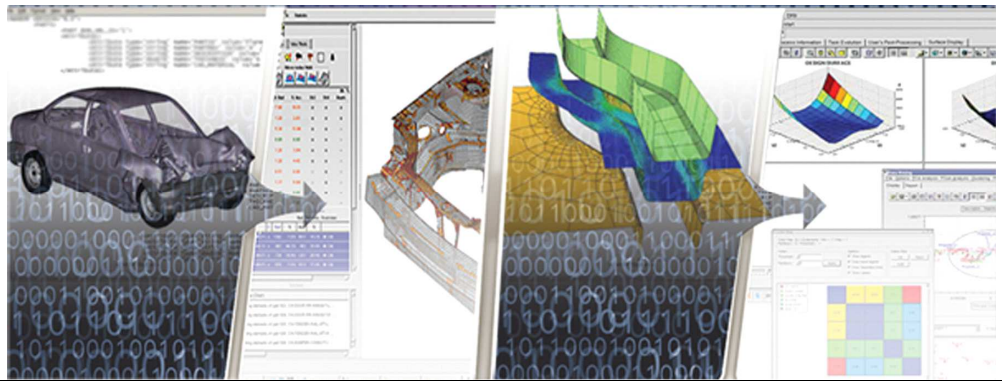

RADIOSS THEORY MANUAL

Version 2017 – January 2017

Large Displacement Finite Element Analysis
Chapter 2

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Chapter 2

BASIC EQUATIONS

2.0 BASIC EQUATIONS

The continuum mechanics summarized here is based on Refs [35], [36] and [37]. Three basic choices need to be made in the development of a large deformation semi discretization scheme:

- the mesh description,
- the kinematic description, i.e. how the deformation is measured,
- the kinetic description, i.e. how the stresses are measured.

Usually, the kinematic description implies the kinetic description as kinetic and kinematic measures should be energetically conjugated.

To go further in to the theory, two sets of coordinates are introduced:

- the spatial or Eulerian coordinates,
- the material or Lagrangian coordinates.

2.1 Material and Spatial Coordinates

In a Cartesian coordinates system, the coordinates of a material point in a reference or initial configuration are denoted by X. The coordinates of the same point in the deformed or final configuration are denoted by x.

The motion or deformation of a body can thus be described by a function $\varphi(X,t)$ where the material coordinates X and the time t are considered as independent variables:

$$x = \varphi(X,t) \qquad \text{EQ. 2.1.0.1}$$

The function φ gives the spatial positions of material points as a function of time.

The displacement of a material point is the difference between its original and final positions:

$$u(X,t) = \varphi(X,t) - X \qquad \text{EQ. 2.1.0.2}$$

It is possible to consider displacements and, as a consequence final coordinates x, as functions of initial coordinates X. The initial configuration is assumed to be perfectly known and each coordinate X identifies a specific material point. For this reason, the initial coordinates are called the material coordinates.

On the other hand, the final coordinates x identify a point of space which can be occupied by different material points according to the different analyzed configurations. For these reasons, the x is called spatial coordinates.

In solid mechanics, material coordinates are usually called Lagrangian coordinates. In their general definition, they are given by the values of the integration constants of the differential equations of particle trajectories. A particular definition consists in using the coordinates X of the particle in the initial configuration. This point of view corresponds to the definition of material coordinates in solid mechanics.

Use of material coordinates is well suited for solid mechanics as we seek to analyze the evolution of a set of points for which we search the final configuration and properties. Integration can be performed in the initial configuration for which geometric properties are usually simple.

In fluid mechanics however, the engineer is more interested in the evolution of a situation in a region defined by fixed boundaries in space. Boundaries are eventually crossed by fluid particles. It is the spatial configuration which is important while the set of particles may vary. This is the reason why fluid mechanics is usually developed using spatial or Eulerian coordinates.

In solid mechanics, the Eulerian formulation consists in considering displacements and initial coordinates as function of spatial coordinates x. A problem for using Eulerian coordinates in solids mechanics is the difficulty of formulating constitutive equations, such as the relationship between stresses and strains that can take into account change of orientation. For this reason solid mechanics are principally developed using the Lagrangian point of view.

The reason for using the Lagrangian form for solids is primarily due to the need for accurate boundary modeling.

2.2 Mesh Description

In *Lagrangian meshes*, mesh points remain coincident with material points and the elements deform with the material. Since element accuracy and time step degrade with element distortion, the magnitude of deformation that can be simulated with Lagrangian meshes is limited.

In *Eulerian meshes*, the coordinates of the element nodes are fixed. This means that the nodes remain coincident with spatial points. Since elements are not changed by the deformation of the material, no degradation in accuracy occurs because of material deformation. On the other hand, in Eulerian meshes, boundary nodes do not always remain coincident with the boundaries of the domain. Boundary conditions must be applied at points which are not nodes. This leads to severe complications in multi-dimensional problems.

A third type of mesh is an *Arbitrary Lagrangian Eulerian* mesh (ALE). In this case, nodes are programmed to move arbitrarily. Usually, nodes on the boundaries are moved to remain on boundaries. Interior nodes are moved to minimize element distortion.

The selection of an appropriate mesh description, whether a Lagrangian, Eulerian or ALE mesh is very important, especially in the solution of the large deformation problems encountered in process simulation or fluid-structure interaction.

A by-product of the choice of mesh description is the establishment of the independent variables. For a Lagrangian mesh, the independent variable is *X*. At a quadrature point used to evaluate the internal forces, the coordinate *X* remains invariant regardless of the deformation of the structure. Therefore, the stress has to be defined as a function of the material coordinate *X*. This is natural in a solid since the stress in a path-dependent material depends on the history observed by a material point. On the other hand, for an Eulerian mesh, the stress will be treated as a function of *x*, which means that the history of the point will need to be convected throughout the computation.

2.3 Vicinity Transformation

Central to the computation of stresses and strains is the Jacobian matrix which relates the initial and deformed configuration:

$$dx_i = \frac{\partial x_i}{\partial X_j} dX_j = D_j x_i dX_j = F_{ij} dX_j \qquad \text{EQ. 2.3.0.1}$$

$$D_j = \frac{\partial}{\partial X_j} \qquad \text{EQ. 2.3.0.2}$$

The transformation is fully described by the elements of the Jacobian matrix *F*:

$$F_{ij} \equiv D_j x_i \qquad \text{EQ. 2.3.0.3}$$

So that EQ. 2.3.0.1 can be written in matrix notation:

$$dx = FdX \qquad \text{EQ. 2.3.0.4}$$

The Jacobian, or determinant of the Jacobian matrix, measures the relation between the initial volume $d\Omega$ and the volume in the initial configuration containing the same points:

$$d\Omega = \left|F\right| d\Omega^0 \qquad \text{EQ. 2.3.0.5}$$

Physically, the value of the Jacobian cannot take the zero value without cancelling the volume of a set of material points. So the Jacobian must not become negative whatever the final configuration. This property insures the existence and uniqueness of the inverse transformation:

$$dX = F^{-1} dx \qquad \text{EQ. 2.3.0.6}$$

At a regular point whereby definition of the field $u(X)$ is differentiable, the vicinity transformation is defined by:

$$F_{ij} = D_j x_i = D_j\left(X_i + u_i(X,t)\right) = \delta_{ij} + D_j u_i \qquad \text{EQ. 2.3.0.7}$$

or in matrix form:

$$F = 1 + A \qquad \text{EQ. 2.3.0.8}$$

So, the Jacobian matrix *F* can be obtained from the matrix of gradients of displacements:

$$A \equiv D_j u_i \qquad \text{EQ. 2.3.0.9}$$

2.4 Kinematic Description

For geometrically non-linear problems, i.e. problems in which rigid body rotations and deformation are large, a large number of measures of deformation are possible but most theoretical work and computer software employ the following three measures:

- the velocity strain (or rate of deformation)

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad \text{EQ. 2.4.0.1}$$

- the Green strain tensor (Lagrangian strain tensor) measured with respect to initial configuration

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right) \qquad \text{EQ. 2.4.0.2}$$

- the Almansi strain tensor (Eulerian strain tensor) measured with respect to deformed configuration

$$E^A{}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) \qquad \text{EQ. 2.4.0.3}$$

All three measures of strains can be related to each other and can be used with any type of mesh.

2.4.1 Velocity strain or rate of deformation

The strain rate is derived from the spatial velocity derivative:

$$\dot{\varepsilon}_{ij} = \frac{d\varepsilon_{ij}}{dt} = D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad \text{EQ. 2.4.1.1}$$

or in matrix form:

$$\dot{\varepsilon} = D = \frac{1}{2}\left(L + L^T\right) \qquad \text{EQ. 2.4.1.2}$$

where:

$$L_{ij} = \frac{\partial v_i}{\partial x_j} \qquad \text{EQ. 2.4.1.3}$$

is the velocity gradient in the current configuration.

The velocity of a material particle is:

$$v_i = \frac{\partial x_i}{\partial t}$$ EQ. 2.4.1.4

where the partial differentiation with respect to time *t* means the rate of change of the spatial position *x* of a given particle. The velocity difference between two particles in the current configuration is given by:

$$dv_i = \frac{\partial v_i}{\partial x_j} dx_j = L_{ij} dx_j = L_{ij} F_{jk} dX_k$$ EQ. 2.4.1.5

In matrix form:

$$dv = Ldx = LFdX$$ EQ. 2.4.1.6

On the other hand, it is possible to write the velocity difference directly as:

$$dv = \frac{\partial}{\partial t}\left(FdX\right) = \dot{F}dX$$ EQ. 2.4.1.7

where:

$$\dot{F} = \frac{\partial F}{\partial t}$$ EQ. 2.4.1.8

One has as a result:

$$L = \dot{F}F^{-1}$$ EQ. 2.4.1.9

Now, *L* is composed of a rate of deformation and a rate of rotation or spin:

$$L = D + \Omega$$ EQ. 2.4.1.10

Since these are rate quantities, the spin can be treated as a vector. It is thus possible to decompose *L* into a symmetric strain rate matrix and an anti symmetric rotation rate matrix just as in the small motion theory the infinitesimal displacement gradient is decomposed into an infinitesimal strain and an infinitesimal rotation. The symmetric part of the decomposition is the strain rate or the rate of deformation and is:

$$\dot{\varepsilon} = D = \frac{1}{2}\left(\dot{F}F^{-1} + F^{-T}\dot{F}^{T}\right)$$ EQ. 2.4.1.11

The anti symmetric part of the decomposition is the spin matrix:

$$\Omega = \frac{1}{2}\left(\dot{F}F^{-1} - F^{-T}\dot{F}^{T}\right)$$ EQ. 2.4.1.12

The velocity-strain measures the current rate of deformation, but it gives no information about the total deformation of the continuum. In general, EQ. 2.4.1.10 is not integral analytically; except in the unidimensional case, where one obtains the true strain:

$$\varepsilon = In\left(l / L\right)$$ EQ. 2.4.1.13

l and *L* are respectively the dimensions in the deformed and initial configurations. Furthermore, the integral in time for a material point does not yield a well-defined, path-independent tensor so that information about phenomena such as total stretching is not available in an algorithm that employs only the strain velocity. Therefore, to obtain a measure of total deformation, the strain velocity has to be transformed to some other strain rate.

The volumetric strain is calculated from density. For one dimensional deformation:

$$\mu = \frac{\rho}{\rho_0} - 1 = \frac{-\delta\Omega}{\Omega} = \frac{-\delta l}{l}$$ EQ. 2.4.1.14

2.4.2 Green strain tensor

The square of the distance which separates two points in the final configuration is given in matrix form by:

$$dx^T dx = dX^T F^T F dX \qquad \text{EQ. 2.4.2.1}$$

Subtracting the square or the initial distance, we have:

$$dx^T dx - dX^T dX = dX^T \left(F^T F - \mathrm{I}\right) dX = 2E dX^T dX \qquad \text{EQ. 2.4.2.2}$$

$$E = \frac{1}{2}\left(F^T F^T - \mathrm{I}\right) \qquad \text{EQ. 2.4.2.3}$$

$C = F^T F$ and $B = FF^T$ are called respectively right and left Cauchy-Green tensor.

Using EQ. 2.3.0.8:

$$E = \frac{1}{2}\left(A + A^T + A^T A\right) \qquad \text{EQ. 2.4.2.4}$$

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right) \qquad \text{EQ. 2.4.2.5}$$

In the unidimensional case, the value of the strain is:

$$E = \left(l^2 - L^2\right)/\left(2L^2\right) \qquad \text{EQ. 2.4.2.6}$$

where *l* and *L* are respectively the dimensions in the deformed and initial configurations.

It can be shown that any motion *F* can always be represented as a pure rigid body rotation followed by a pure stretch of three orthogonal directions:

$$F = RU = R(I + H) \qquad \text{EQ. 2.4.2.7}$$

with the rotation matrix *R* satisfying the orthogonality condition:

$$R^T R = I \qquad \text{EQ. 2.4.2.8}$$

and *H* symmetric.

The polar decomposition theorem is important because it will enable to distinguish the straining part of the motion from the rigid body rotation.

One has from EQS. 2.4.2.3 and 2.4.2.7:

$$E = H + \frac{H^2}{2} = \frac{1}{2}\left(F^T F - I\right) \qquad \text{EQ. 2.4.2.9}$$

$$R = F\left(I + H\right)^{-1} \qquad \text{EQ. 2.4.2.10}$$

EQ. 2.4.2.9 allows the computation of *H*, and EQ. 2.4.2.10. of *R*.

As the decomposition of the Jacobian matrix *F* exists and is unique, *H* is a new measure of strain which is sometimes called the Jaumann strain. Jaumann strain requires the calculation of principal directions.

If rotations are small,

$$R = I + \Omega \qquad \text{EQ. 2.4.2.11}$$

$$R^T R = \left(I + \Omega\right)^T \left(I + \Omega\right) \qquad \text{EQ. 2.4.2.12}$$

$$\Omega^T + \Omega = 0 \qquad \text{EQ. 2.4.2.13}$$

if second order terms are neglected.

As a result, one has for the Jacobian matrix:

$$F = I + A = (I + \Omega)(I + H)$$ EQ. 2.4.2.14

leading, if the second order terms are neglected, to the classical linear relationships:

$$A = \Omega + H$$ EQ. 2.4.2.15

$$H = \frac{1}{2}\left(A^T + A\right)$$ EQ. 2.4.2.16

$$\Omega = \frac{1}{2}\left(A - A^T\right)$$ EQ. 2.4.2.17

So for EQ. 2.4.2.15 and EQ. 2.4.2.16, when rigid body rotations are large, the linear strain tensor becomes non-zero even in the absence of deformation.

The preceding developments show that the linear strain measure is appropriate if rotations can be neglected; that means if they are of the same magnitude as the strains and if these are of the order of 10^{-2} or less. It is also worth noticing that linear strains can be used for moderately large strains of the order of 10^{-1} provided that the rotations are small. On the other hand, for slender structures which are quite in extensible, non-linear kinematics must be used even when the rotations are order of 10^{-2} because, if we are interested in strains of $10^{-3} - 10^{-4}$, using linear strain the error due to the rotations would be greater than the error due to the strains.

Large deformation problems in which non-linear kinematics is necessary, are those in which rigid body rotation and deformation are large.

2.5 Kinetic Description

The virtual power principle in Section 2.10 will state equilibrium in terms of Cauchy true stresses and the conjugate virtual strain rate, the rate of deformation. It is worth noticing that, from the engineer's point of view, the Cauchy true stress is probably the only measure of practical interest because it is a direct measure of the traction being carried per unit area of any internal surface in the body under study. This is the reason why RADIOSS reports the stress as the Cauchy stress. The second Piola-Kirchhoff stress is, however, introduced here because it is frequently mentioned in standard textbooks.

The relationship between the Piola-Kirchhoff stress and the Cauchy stress is obtained as follows. Starting from the definition of Green's strain (EQ. 2.4.2.3),

$$E = \frac{1}{2}\left(F^T F - I\right)$$ EQ. 2.5.0.1

the strain rate is given by:

$$\dot{E} = \frac{1}{2}\left(\dot{F}^T F + F^T \dot{F}\right)$$ EQ. 2.5.0.2

The power per unit reference volume is:

$$P = \dot{E} S$$ EQ. 2.5.0.3

where *S* represents the tensor of second Piola-Kirchhoff stresses. On the other hand for Cauchy stresses:

$$P = \dot{\varepsilon}\sigma|F|$$ EQ. 2.5.0.4

$$\left(\dot{F}^T F + F^T F\right)S = \left(\dot{F}F^{-1} + F^{-T}\dot{F}^T\right)\sigma|F|$$ EQ. 2.5.0.5

One has immediately:

$$FSF^T = \sigma|F|$$ EQ. 2.5.0.6

Second Piola-Kirchhoff stresses have a simple physical interpretation. They correspond to a decomposition of forces in the frame coordinate systems convected by the deformation of the body. However, the stress measure is still performed with respect to the initial surface.

2.6 Stress Rates

In practice, the true stress (or Cauchy stress) for any time interval will be computed using the stress rate in an explicit time integration:

$$\sigma_{ij}(t+\delta t) = \sigma_{ij}(t) + \dot{\sigma}_{ij}\delta t \qquad \text{EQ. 2.6.0.1}$$

$\dot{\sigma}_{ij}$ is not simply the time derivative of the Cauchy stress tensor as Cauchy stress components are associated with spatial directions in the current configuration. So, the derivatives will be nonzero in the case of a pure rigid body rotation, even if from the constitutive point of view the material is unchanged. The stress rate is a function of element average rigid body rotation and of strain rate.

For this reason, it is necessary to separate $\dot{\sigma}_{ij}$ into two parts; one related to the rigid body motion and the remainder associated with the rate form of the stress-strain law. Objective stress rate is used, meaning that the stress tensor follows the rigid body rotation of the material [14].

A stress law will be objective if it is independent of the space frame. To each definition of the rigid body rotation, corresponds a definition of the objective stress rate. The Jaumann objective stress tensor derivative will be associated with the rigid body rotation defined in EQ. 2.4.1.11:

$$\dot{\sigma}^{v}{}_{ij} = \dot{\sigma}_{ij} - \dot{\sigma}^{r}{}_{ij} \qquad \text{EQ. 2.6.0.2}$$

where:

$\dot{\sigma}^{v}{}_{ij}$ is the Jaumann objective stress tensor derivative,

$\dot{\sigma}^{r}{}_{ij}$ is the stress rate due to the rigid body rotational velocity.

The correction for stress rotation is given by:

$$\dot{\sigma}^{r}{}_{ij} = \sigma_{ik}\Omega_{kj} + \sigma_{jk}\Omega_{ki} \qquad \text{EQ. 2.6.0.3}$$

and Ω_{kj} defined in EQ. 2.4.1.11 (see Section 5.1.10.1).

2.7 Stresses in Solids

2.7.1 Principal stresses

Since the stress tensor is symmetric, we can always find a proper orthogonal matrix, i.e. a coordinate system that diagonalizes it:

$$R^{T}\sigma R = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \qquad \text{EQ. 2.7.1.1}$$

The diagonal components are called the principal stresses and allow a 3D representation of the state of stress at a point.

2.7.2 Stress invariants

Many of the constitutive models in RADIOSS are formulated in terms of invariants of the stress tensor. The most important are the first and second invariants:

$$p = -\frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3}$$ EQ. 2.7.2.1

$$\sigma_{vm} = \sqrt{\frac{3}{2}\left((\sigma_{xx} + p)^2 + (\sigma_{yy} + p)^2 + (\sigma_{zz} + p)^2 + 2\sigma^2{}_{xy} + 2\sigma^2{}_{zy} + 2\sigma^2{}_{xz}\right)}$$ EQ. 2.7.2.2

called pressure and von Mises stress after Richard von Mises.

The values of these functions remain invariant under transformation by a proper orthogonal matrix.

If,

$\sigma = R^{\mathrm{T}}\sigma_0 R$,

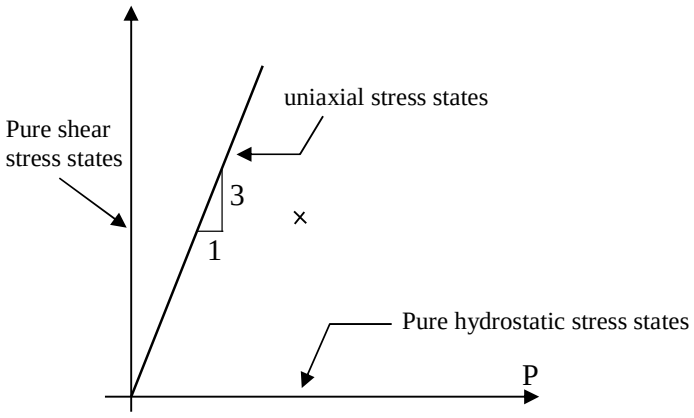
then:

$p = p_0$

$\sigma_{vm} = \sigma_{vm0}$

2.7.3 Invariant space

It is useful to plot the state of stress as a point in a diagram of pressure and von Mises stress:



The horizontal axis corresponds to the hydrostatic loading, the vertical axes to pure shear. The line with tangent 1/3 is uniaxial compression. The line with tangent -1/3 is uniaxial tension.

2.7.4 Deviatoric stresses

The pressure or first invariant is related to the change in volume of the solid. The deviation from a hydrostatic state of stress is linked to the change in shape. The stress deviator is defined as:

$S = \sigma + pI$

The second invariant becomes, in terms of the deviators:

$$\sigma_{vm} = \sqrt{\frac{3}{2}\left(s^2{}_{xx} + s^2{}_{yy} + s^2{}_{zz} + 2s^2{}_{xy} + 2s^2{}_{yz} + 2s^2{}_{xz}\right)}$$ EQ. 2.7.4.1

A surface of constant von Mises stress in deviatoric space or principal deviatoric space is a sphere (in stress space it is a cylinder).

2.8 Updated, Total Lagrangian and Corotational Formulations

Finite element discretizations with Lagrangian meshes are commonly classified as either an updated Lagrangian formulation or a total Lagrangian formulation. Both formulations use a Lagrangian description. That means that the dependent variables are functions of the material (Lagrangian) coordinates and time. In the geometrically nonlinear structural analysis the configuration of the structure must be tracked in time. This tracking process necessary involves a kinematic description with respect to a reference state. Three choices called “kinematic descriptions” have been extensively used:

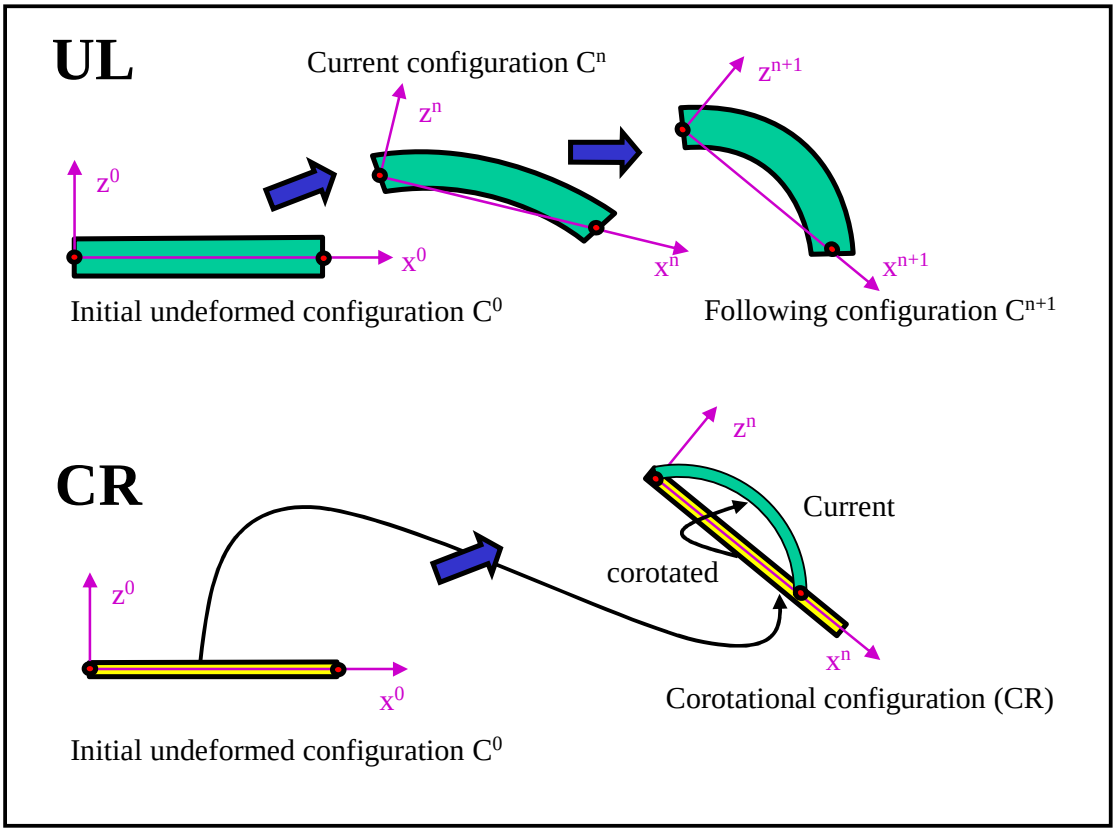
Total Lagrangian description (TL): The FEM equations are formulated with respect to a fixed reference configuration which is not changed throughout the analysis. The initial configuration is often used; but in special cases the reference could be an artificial base configuration.

Updated Lagrangian description (UL): The reference is the last known (accepted) solution. It is kept fixed over a step and updated at the end of each step.

Corotational description (CR): The FEM equations of each element are referred to two systems. A fixed or base configuration is used as in TL to compute the rigid body motion of the element. Then the deformed current state is referred to the corotated configuration obtained by the rigid body motion of the initial reference.

The updated Lagrangian and corotational formulations are the approaches used in RADIOSS. These two approaches are schematically presented in Figure 2.8.1.

Figure 2.8.1 Updated Lagrangian and Corotational descriptions



By default, RADIOSS uses a large strain, large displacement formulation with explicit time integration. The large displacement formulation is obtained by computing the derivative of the shape functions at each cycle. The large strain formulation is derived from the incremental strain computation. Hence, stress and strains are true stresses and true strains.

In the updated Lagrangian formulation, the Lagrangian coordinates are considered instantaneously coincident with the Eulerian spatial *x* coordinates. This leads to the following simplifications:

$$\frac{\partial x_i}{\partial X_j} = \frac{\partial X_j}{\partial x_i} = \delta_{ij} \qquad \text{EQ. 2.8.0.1}$$

$$d\Omega = d\Omega_0 \qquad \text{EQ. 2.8.0.2}$$

The derivatives are with respect to the spatial (Eulerian) coordinates. The weak form involves integrals over the deformed or current configuration. In the total Lagrangian formulation, the weak form involves integrals over the initial (reference) configuration and derivatives are taken with respect to the material coordinates.

The corotational kinematic description is the most recent of the formulations in geometrically nonlinear structural analysis. It decouples small strain material nonlinearities from geometric nonlinearities and handles naturally the question of frame indifference of anisotropic behavior due to fabrication or material nonlinearities. Several important works outline the various versions of CR formulation [7], [50], [51], [52], and [53].

Some new generation of RADIOSS elements are based on this approach. Refer to the "Element Library" chapter for more details.

REMARK:

A similar approach to CR description using convected-coordinates is used in some branches of fluid mechanics and theology. However, the CR description maintains orthogonality of the moving frames. This will allow achieving an exact decomposition of rigid body motion and deformational modes. On the other hand, convected coordinates form a curvilinear system that fits the change of metric as the body deforms. The difference tends to disappear as the mesh becomes finer. However, in general case the CR approach is more convenient in structural mechanics.

2.9 Equations of Equilibrium

Let Ω be a volume occupied by a part of the body in the current configuration, and Γ the boundary of the body. In the Lagrangian formulation, Ω is the volume of space occupied by the material at the current time, which is different from the Eulerian approach where we examine a volume of space through which the material passes. τ is the traction surface on Γ and *b* are the body forces.

Force equilibrium for the volume is then:

$$\int_\Gamma \tau_i d\Gamma + \int_\Omega \rho b_i d\Omega = \int_\Omega \rho \frac{\partial v_i}{\partial t} d\Omega \qquad \text{EQ. 2.9.0.1}$$

with ρ the material density.

The Cauchy true stress matrix at a point of Γ is defined by:

$$\tau_i = n_j \sigma_{ji} \qquad \text{EQ. 2.9.0.2}$$

Where, *n* is the outward normal to Γ at that point. Using this definition, EQ. 2.9.0.1 is written:

$$\int_\Gamma n_j \sigma_{ji} d\Gamma + \int_\Omega \rho b_i d\Omega = \int_\Omega \rho \frac{\partial v_i}{\partial t} d\Omega \qquad \text{EQ. 2.9.0.3}$$

Gauss' theorem allows the rewrite of the surface integral as a volume integral so that:

$$\int_{\Gamma} n_j \sigma_{ji} d\Gamma = \int_{\Omega} \frac{\partial \sigma_{ij}}{\partial x_j} d\Omega \qquad \text{EQ. 2.9.0.4}$$

As the volume is arbitrary, the expression can be applied at any point in the body providing the differential equation of translation equilibrium:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho \frac{\partial v_i}{\partial t} \qquad \text{EQ. 2.9.0.5}$$

Use of Gauss' theorem with this equation leads to the result that the true Cauchy stress matrix must be symmetric:

$$\sigma = \sigma^T \qquad \text{EQ. 2.9.0.6}$$

so that at each point there are only six independent components of stress. As a result, moment equilibrium equations are automatically satisfied, thus only the translational equations of equilibrium need to be considered.

2.10 Principle of Virtual Power

The basis for the development of a displacement finite element model is the introduction of some locally based spatial approximation to parts of the solution. The first step to develop such an approximation is to replace the equilibrium equations by an equivalent weak form. This is obtained by multiplying the local differential equation by an arbitrary vector valued test function defined with suitable continuity over the entire volume and integrating over the current configuration:

$$\int_{\Omega} \left(\delta v_i \left(\frac{\partial \sigma_{ji}}{\partial x_j} + \rho b_i - \rho \dot{v}_i \right) \right) d\Omega = 0 \qquad \text{EQ. 2.10.0.1}$$

The first term in EQ. 2.10.0.1 is then expanded:

$$\int_{\Omega} \left(\delta v_i \frac{\partial \sigma_{ji}}{\partial x_j} \right) d\Omega = \int_{\Omega} \left[\frac{\partial}{\partial x_j} \left((\delta v_i) \sigma_{ji} \right) - \frac{\partial (\delta v_i)}{\partial x_j} \sigma_{ji} \right] d\Omega \qquad \text{EQ. 2.10.0.2}$$

Using Gauss's theorem gives:

$$\int_{\Omega} \left(\frac{\partial}{\partial x_j} \left((\delta v_i) \sigma_{ji} \right) \right) d\Omega = \int_{\Gamma_\sigma} \left[(\delta v_i) n_j \sigma_{ji} \right] d\Gamma \qquad \text{EQ. 2.10.0.3}$$

taking into account that stresses vanish on the complement of the traction boundaries.

Replacing EQ. 2.10.0.3 in EQ. 2.10.0.2 gives:

$$\int_{\Omega} \left(\delta v_i \frac{\partial \sigma_{ji}}{\partial x_j} \right) d\Omega = \int_{\Gamma} (\delta v_i) \tau_i d\Gamma - \int_{\Omega} \frac{\partial (\delta v_i)}{\partial x_j} \sigma_{ji} d\Omega \qquad \text{EQ. 2.10.0.4}$$

If this last equation is then substituted in EQ. 2.10.0.1, one obtains:

$$\int_{\Omega} \left(\frac{\partial (\delta v_i)}{\partial x_j} \right) \sigma_{ji} d\Omega - \int_{\Omega} \delta v_i \rho b_i d\Omega - \int_{\Gamma} (\delta v_i) \tau_i d\Gamma + \int_{\Omega} \delta v_i \rho \dot{v}_i d\Omega = 0 \qquad \text{EQ. 2.10.0.5}$$

The preceding expression is the weak form for the equilibrium equations, traction boundary conditions and interior continuity conditions. It is known as the principle of virtual power.

2.11 Physical Names of Virtual Power Terms

It is possible to give a physical name to each of the terms in the virtual power equation. This will be useful in the development of finite element equations. The nodal forces in the finite element equations will be identified according to the same physical names.

The first term can be successively written:

$$\frac{\partial(\delta v_i)}{\partial x_j}\sigma_{ji} = (\delta L_{ij})\sigma_{ji} = (\delta D_{ij} + \delta W_{ij})\sigma_{ji} = \delta D_{ij}\sigma_{ji} \qquad \text{EQ. 2.11.0.1}$$

One has used the decomposition of the velocity gradient *L* into its symmetric and skew symmetric parts and that $\delta W_{ij}\sigma_{ji} = 0$ since δW_{ij} is skew symmetric and σ_{ji} is symmetric.

The latter relation suggests that $\delta D_{ij}\sigma_{ji}$ can be interpreted as the rate of internal virtual work or virtual internal power per unit volume. The total internal power δP^{int} is defined by the integral of $\delta D_{ij}\sigma_{ji}$:

$$\delta P^{\text{int}} = \int_\Omega \delta D_{ij}\sigma_{ji}\, d\Omega = \int_\Omega \frac{\partial(\delta v_i)}{\delta x_j}\sigma_{ji}\, d\Omega \equiv \int_\Omega \delta L_{ij}\sigma_{ji}\, d\Omega \qquad \text{EQ. 2.11.0.2}$$

The second and third terms in EQ. 2.10.0.5 are the virtual external power:

$$\delta P^{ext} = \int_\Omega \delta v_i \rho b_i\, d\Omega + \int_{\Gamma_\sigma} (\delta v_i)\tau_i\, d\Gamma \qquad \text{EQ. 2.11.0.3}$$

The last term is the virtual inertial power:

$$\delta P^{inert} = \int \delta v_i \rho \dot{v}_i\, d\Omega \qquad \text{EQ. 2.11.0.4}$$

Inserting EQS 2.11.0.2, 2.11.0.3 and 2.11.0.4 into EQ. 2.11.0.5, the principle of virtual power can be written as:

$$\delta P = \delta P^{\text{int}} - \delta P^{ext} + \delta P^{inert} \qquad \text{EQ. 2.11.0.5}$$

for all δv_i admissible.

We can show that virtual power principle implies strong equations of equilibrium. So it is possible to use the virtual power principle with a suitable test function as a statement of equilibrium.

The virtual power principle has a simple physical interpretation. The rate of work done by the external forces subjected to any virtual velocity field is equal to the rate of work done by the equilibrating stresses on the rate of deformation of the same virtual velocity field. The principle is the weak form of the equilibrium equations and is used as the basic equilibrium statement for the finite element formulation. Its advantage in this regard is that it can be stated in the form of an integral over the volume of the body. It is possible to introduce approximations by choosing test functions for the virtual velocity field whose variation is restricted to a few nodal values.

2.12 Small Strain Formulation

RADIOSS uses two different methods to calculate stress and strain. The method used depends on the type of simulation. The two types are:

- Large strain
- Small strain

The large strain formulation has been discussed before and is used by default. Small strain analysis is best used when the deformation is known to be small, for example, linear elastic problems.

Large strain is better suited to non-linear, elastoplastic behavior where large deformation is known to occur. However, large mesh deformation and distortion can create problems with the time step. If an element is deformed excessively, the time step will decrease too much, increasing the CPU time. If the element reaches a

negative volume, the computation will stop or the element will have to be removed. Using small strain can eliminate these problems.

Using a small strain formulation for part of a large deformation process introduces of course errors. These errors depend on the specific case, but they can provide a better solution than element deletion.

On the other side, materials like honeycomb, which have no Poisson's effect, can have the small strain limitations corrected by using adjusted stress-strain curves.

A small strain, small displacement formulation can thus be specified for some specific material behavior, like honeycomb, or can be implemented when the time step with a large strain formulation reaches a minimum value that is defined by the user. This allows the computation to proceed at an acceptable rate.

The small displacement formulation is, however, not recommended for some simulations, e.g. crash analysis.

2.12.1 Small strain option

Assuming a constant Jacobian matrix during time and also a constant volume, previous equations degenerate into a small strain and small displacement formulation. All spatial variables are then values defined at time t=0 (or at the time the small strain formulation is initiated).

Time step then becomes constant:

$$\Delta t = \frac{l_0}{c} \qquad \text{EQ. 2.12.1.1 (a)}$$

and the effective negative volume has no effect on the computation (only the initial volume is used).

The Jacobian matrix time transformation is dependent upon element deformation and element rigid body rotation (EQ. 2.4.2.7 and EQ. 2.4.2.8). On the other hand, rigid body translation has no effect on the Jacobian matrix.

A small strain formulation is achieved if the element deformation is not taken into account. Likewise, a small displacement formulation is obtained if the element rigid body rotation is ignored.

From a practical point of view, small strain formulation will be obtained if, instead of recomputing the Jacobian matrix at each cycle, the initial matrix is updated taking into account element rigid body rotation:

$$F(t+\delta t) = F(t)\Omega \qquad \text{EQ. 2.12.1.1 (b)}$$

where Ω is the rigid body angular velocity.

An alternative solution that accounts for element rigid body rotation consists in computing the internal forces in a local reference frame attached to the element. This solution is used for shell elements and convected brick elements.

Unlike the large strain formulation, the small strain formulation uses values based on the initial configuration. This is either at the beginning of the simulation or at the beginning of the small strain implementation.

Hence, the strain rate is calculated using:

$$\dot{\varepsilon}_{ij} = \left(\frac{\partial \Phi_I}{\partial x_j} \right)_{t=0} v_{iI} \qquad \text{EQ. 2.12.1.2}$$

with Φ_I the interpolating shape functions and v_{iI} the components of velocity at node *I*.

The strain in an arbitrary *x* direction is calculated by:

$$\varepsilon_x = \sum \dot{\varepsilon}_x dt = \sum \left(\frac{\Delta \delta x}{x_0} \right) = \frac{\Delta x}{x_0} \qquad \text{EQ. 2.12.1.3}$$

Thus, the strain is the engineering strain.

The stress is calculated using the strain rate and the material law provided by the user. The later is integrated over the element volume to produce the internal force vector, which is summed over the elements to obtain the overall force vector:

$$f_{iI}^{\text{int}} = \int_{\Omega} \sigma_{ij} \left(\frac{\partial \phi_I}{\partial x_j} \right)_{t=0} d\Omega$$ EQ. 2.12.1.4

The stress is the engineering stress.

The volumetric strain using the small strain formulation is independent of density. For one dimensional deformation, one has:

$$\mu = -\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right) = -\frac{\delta l}{l_0}$$ EQ. 2.12.1.5

The small strain formulation for solid elements was developed for specific material, like honeycomb. In the crushing direction, honeycomb has no Poisson's effect and stress integration over the initial surface is acceptable. The effect on strain is small during elastic deformation and can be corrected in the plastic phase by using a modified engineering stress-engineering strain material curve.

For materials like crushable foam, with a small Poisson's ratio, this formulation can be applied successfully in certain situations. However, for other materials, this formulation has to be used very carefully.

Shell elements have fewer limitations than solid elements. For crash applications, the main shell deformation is bending. The small strain formulation has no effect on the bending description if membrane deformation is small.

The small strain formulation can be applied to some elements for which the time step is reaching a user specified value.

If the critical time step is small, compared to the initial one, this formulation gives acceptable results and is more accurate than removing the deformed elements.

2.12.2 Large strain option

By default RADIOSS uses a large strain large displacement formulation with explicit time integration. By computing the derivative of shape functions at each cycle, large displacement formulation is obtained. The large strain formulation results from incremental strain computation. Stresses and strains are therefore true stresses and true strains.

The spatial derivatives of isoparametric brick shape functions are given by:

$$\frac{\partial \Phi_I}{\partial x_j} = F(t)^{-1} \frac{\partial \Phi_I}{\partial r}$$ EQ. 2.12.2.1

Where *F(t)* is the Jacobian matrix.

For each element the internal forces are integrated over the volume with one integration point:

$$f_{iI}^{\text{int}} = \int_{\Omega} \left(\sigma_{ij} \left(\frac{\partial \Phi_I}{\partial x_i} \right)_{t=0} d\Omega = \sigma_{ij} \frac{\partial \Phi_I}{\partial x_i} \Omega \right)$$ EQ. 2.12.2.2

Time integration of Cauchy stress (true stress):

$$\sigma_{ij}(t + \delta t) = \sigma_{ij}(t) + \frac{d\sigma_{ij}(t)}{dt} dt$$ EQ. 2.12.2.3

uses objective stress rate, meaning that the stress tensor follows the rigid body rotation of the material. Stress rate is a function of element average rigid body rotation and of strain rate. Strain rate is obtained from spatial velocity derivative:

$$\frac{d\varepsilon_{ij}}{dt} = \frac{1}{2} \left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right]$$ EQ. 2.12.2.4

where:

$$\frac{\partial v_i}{\partial x_j} = \frac{\partial \Phi_I}{\partial x_j v_i}$$ EQ. 2.12.2.5

Stability of explicit scheme is given by the Courant condition:

$$\Delta t < \frac{l}{c}$$ EQ. 2.12.2.6

with *l* the element characteristic length and *c* the sound speed.

The time step is computed at each cycle.

Large element deformation can give a large time step decrease. For overly large deformations a negative volume can be reached and it then becomes impossible to invert the Jacobian matrix and to integrate the stresses over the volume.

2.12.3 Stress and strain definition

With large strain formulation, stresses are true stresses and strains are true strains:

$$\varepsilon = \sum \Delta\Delta l / l \equiv \ln \frac{l}{l_0}$$ EQ. 2.12.3.1

$$\sigma = \frac{F}{S}$$ EQ. 2.12.3.2

With small strain formulation stresses become engineering stresses and strains engineering strains:

$$\varepsilon = \sum \Delta\Delta l / l_0 \equiv \frac{\Delta l}{l_0}$$ EQ. 2.12.3.3

$$\sigma = \frac{F}{S_0}$$ EQ. 2.12.3.4

The definition of volumic strain is also modified. For large strain RADIOSS uses a volumic strain computed from density:

$$\mu = \left(\frac{\rho}{\rho_0} - 1 \right) = \frac{\Delta V}{V} = \frac{\Delta l}{l}$$ EQ. 2.12.3.5

For small strain we have:

$$\mu = -\left(\varepsilon_x + \varepsilon_y + \varepsilon_z \right) = -\frac{\Delta l}{l_0}$$ EQ. 2.12.3.6