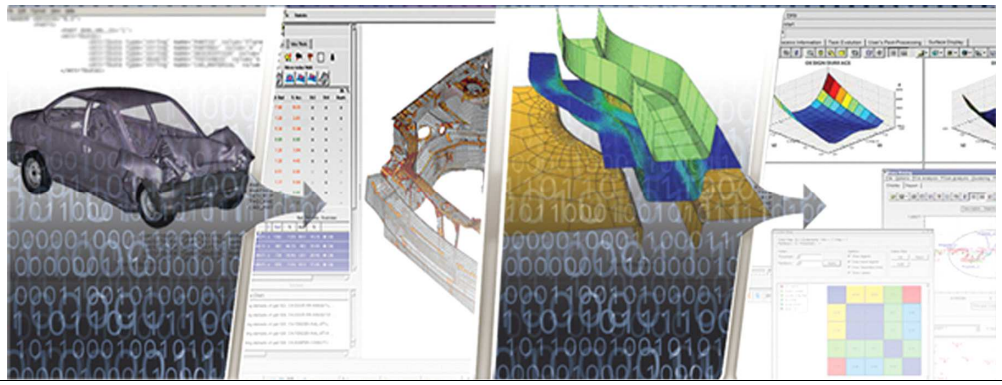

RADIOSS THEORY MANUAL

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Large Displacement Finite Element Analysis
Chapter 4

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Chapter 4

DYNAMIC ANALYSIS

4.0 DYNAMIC ANALYSIS

The discrete form of the equation of motion given in Section 3.5 represents a system of linear differential equations of second order and, in principal the solution to the equations can be obtained by standard procedures for the solution of differential equations with constant coefficients. However, in practical finite element analysis, a few effective methods are used. The procedures are generally divided into two methods of solution: direct integration method and mode superposition. Although the two techniques may at first look to be quite different, in fact they are closely related, and the choice for one method or the other is determined only by their numerical effectiveness.

In *direct integration* the equations of motion are directly integrated using a numerical step-by-step procedure. In this method no transformation of the equations into another basis is carried out. The dynamic equilibrium equation written at discrete time points includes the effect of inertia and damping forces. The variation of displacements, velocities and accelerations is assumed with each time interval Δt. As the solution is obtained by a step-by-step procedure, the diverse system nonlinearities as geometric, material, contact and large deformation nonlinearity are taken into account in a natural way even if the resolution in each step remains linear.

The *mode superposition* method generally consists of transforming the equilibrium equation into the *generalized displacement* modes. An eigen value problem is resolved. The eigen vectors are the free vibration mode shapes of the finite element assemblage. The superposition of the response of each eigen vector leads to the global response. As the method is based on the superposition rule, the linear response of dynamically loaded of the structure is generally developed.

In the following, first the resolution procedure in direct integration method when using an explicit time discretization scheme is described. Then, the procedures of modal analysis are briefly presented. The implicit method will be detailed in Chapter 12.

4.1 Direct Integration Method via Explicit Scheme

In transient dynamic analysis, the direct integration method is usually chosen. A few commonly used integration methods exist in the literature [54]. The method used in RADIOSS is derived from Newmark time integration scheme.

This section deals with time integration of accelerations, velocities and displacements. The general algorithm for computing accelerations, velocities and displacements is given. Stability and time step aspects are then discussed.

4.1.1 Newmark's Method

Newmark's method is a one step integration method. The state of the system at a given time $t_{n+1} = t_n + h$ is computed using Taylor's formula:

$$f(t_n + h) = f(t_n) + hf'(t_n) + \frac{h^2}{2} f^{(2)}(t_n) + \ldots + \frac{h^s}{s!} f^{(s)}(t_n) + R_s \qquad \text{EQ. 4.1.1.1}$$

$$R_s = \frac{1}{s!} \int_{t_n}^{t_n+h} f^{(s+1)}(\tau)[t_n + h - \tau]^s \, d\tau \qquad \text{EQ. 4.1.1.2}$$

The preceding formula allows the computation of displacements and velocities of the system at time $t_n + 1$:

$$\dot{u}_{n+1} = \dot{u}_n + \int_{t_n}^{t_{n+1}} \ddot{u}(\tau) d\tau \qquad \text{EQ. 4.1.1.3}$$

$$u_{n+1} = u_n + h\dot{u}_n + \int_{t_n}^{t_{n+1}} (t_{n+1} - \tau)\ddot{u}(\tau)d\tau$$ EQ. 4.1.1.4

The approximation consists in computing the integrals for acceleration in 4.1.1.3 and in 4.1.1.4 by numerical quadrature:

$$\int_{t_n}^{t_{n+1}} \ddot{u}(\tau)d\tau = (1-\gamma)h\ddot{u}_n + \gamma h\ddot{u}_{n+1} + r_n$$ EQ. 4.1.1.5

$$\int_{t_n}^{t_{n+1}} (t_{n+1} - \tau)\ddot{u}(\tau)d\tau = \left(\frac{1}{2} - \beta\right)h^2\ddot{u}_n + \beta h^2\ddot{u}_{n+1} + r_n'$$ EQ. 4.1.1.6

In replacing EQ. 4.1.1.3 and EQ. 4.1.1.4, one has:

$$\dot{u}_{n+1} = \dot{u}_n + (1-\gamma)h\ddot{u}_n + \gamma h\ddot{u}_{n+1}$$ EQ. 4.1.1.7

$$u_{n+1} = u_n + h\dot{u}_n + \left(\frac{1}{2} - \beta\right)h^2\ddot{u}_n + \beta h^2\ddot{u}_{n+1} + r_n'$$ EQ. 4.1.1.8

According to the values of γ and β, different algorithms can be derived:

- $\gamma = 0, \beta = 0$: pure explicit algorithm. It can be shown that it is always unstable. An integration scheme is stable if a critical time step exists so that, for a value of the time step lower or equal to this critical value, a finite perturbation at a given time does not lead to a growing modification at future time steps.
- $\gamma = 1/2, \beta = 0$: central difference algorithm. It can be shown that it is conditionally stable.
- $\gamma = 1/2, \beta = 1/2$: Fox & Goodwin algorithm.
- $\gamma = 1/2, \beta = 1/6$: linear acceleration.
- $\gamma = 1/2, \beta = 1/4$: mean acceleration. This integration scheme is the unconditionally stable algorithm of maximum accuracy.

4.1.2 The Central Difference Algorithm

The central difference algorithm corresponds to the Newmark algorithm with $\gamma = \frac{1}{2}$ and $\beta = 0$ so that EQS. 4.1.1.7 and 4.1.1.8 become:

$$\dot{u}_{n+1} = \dot{u}_n + \frac{1}{2}h_{n+1}(\ddot{u}_n + \ddot{u}_{n+1})$$ EQ. 4.1.2.1

$$u_{n+1} = u_n + h_{n+1}\dot{u}_n + \frac{1}{2}h_{n+1}^2 * \ddot{u}_n$$ EQ. 4.1.2.2

with h_{n+1} the time step between t_n and t_{n+1}.

It is easy to show [38] that the central difference algorithm can be changed to an equivalent form with 3 time steps:

$$\ddot{u}_n = \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2}$$ EQ. 4.1.2.3

if the time step is constant.

From the algorithmic point of view, it is, however, more efficient to use velocities at half of the time step:

$$\dot{u}_{n+\frac{1}{2}} = \dot{u}\left(t_{n+\frac{1}{2}}\right) = \frac{1}{h_{n+1}}\left(u_{n+1} - u_n\right)$$ EQ. 4.1.2.4

so that:

$$\ddot{u}_n = \frac{1}{h_{n+\frac{1}{2}}}\left(\dot{u}_{n+\frac{1}{2}} - \dot{u}_{n-\frac{1}{2}}\right)$$ EQ. 4.1.2.5

$$h_{n+\frac{1}{2}} = \left(h_n + h_{n+1}\right)/2$$ EQ. 4.1.2.6

Time integration is *explicit*, in that if acceleration $\ddot{u}_n$ is known (see Section 4.3), the future velocities and displacements are calculated from past (known) values in time:

- $\dot{u}_{n+\frac{1}{2}}$ is obtained from EQ. 4.1.2.5:

$$\dot{u}_{n+\frac{1}{2}} = \dot{u}_{n-\frac{1}{2}} + h_{n+\frac{1}{2}}\ddot{u}_n$$ EQ. 4.1.2.7

The same formulation is used for rotational velocities.

- u_{n+1} is obtained from EQ. 4.1.2.4:

$$u_{n+1} = u_n + h_{n+1}\dot{u}_{n+\frac{1}{2}}$$ EQ. 4.1.2.8

The accuracy of the scheme is of h^2 order, i.e. if the time step is halved, the amount of error in the calculation is one quarter of the original. The time step *h* may be variable from one cycle to another. It is recalculated after internal forces have been computed.

4.1.3 Numerical Starting Procedure

At time *t = 0*, the displacement *u* and velocity $\dot{u}_0$ are known from initial conditions. The acceleration $\ddot{u}_0$ and time step h_1 are found from solving the equations of motion. The initial time step h_0 is set to zero:

$$h_0 = 0 \ ; \qquad h_{\frac{1}{2}} = \frac{h_1}{2}$$ EQ. 4.1.3.1

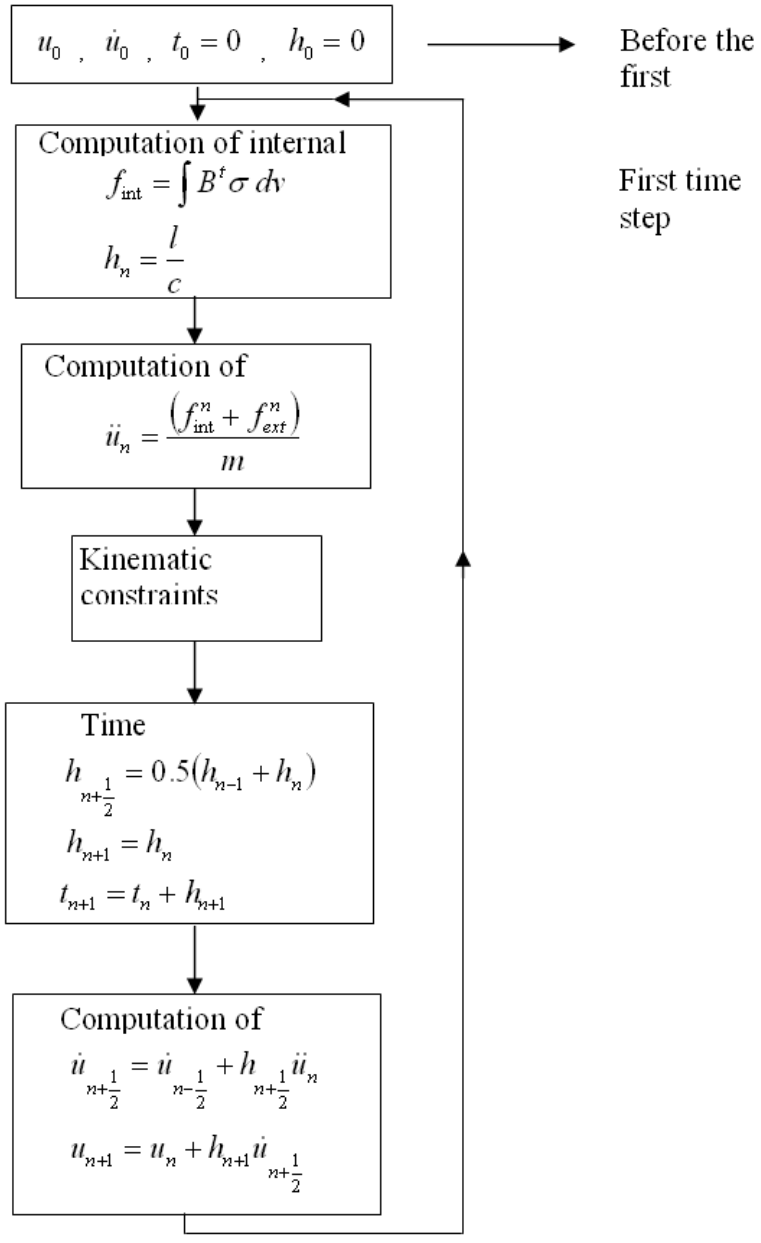
$$\dot{u}_{-\frac{1}{2}} = \dot{u}_0$$ EQ. 4.1.3.2

$$\dot{u}_{\frac{1}{2}} = \dot{u}_0 + h_{\frac{1}{2}}\ddot{u}_0$$ EQ. 4.1.3.3

4.1.4 Algorithm Flow Chart

The flow chart of the central difference algorithm can be summarized as in Fig. 4.1.1. It is pointed out that the solution of the linear system to compute accelerations is immediate if the mass matrix is diagonal.

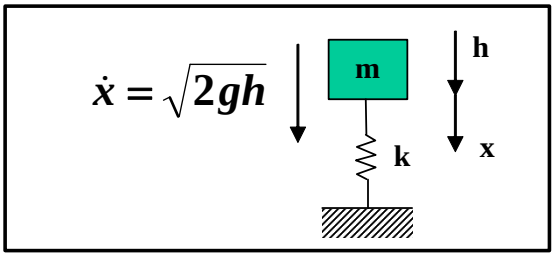
Figure 4.1.1 Numerical Procedure



4.1.5 Example: Dropping body

The question “how far can a body be dropped without incurring damage?” is frequently asked in the packaging manufacturing for transportation of particles. The problem is similar in landing of aircrafts. It can be studied by an analytical approach where the dropping body is modeled by a simple mass-spring system (Figure 4.1.2). If *h* is the dropping height, *m* and *k* the mass of the body and the stiffness representing the contact between the body and the ground, the equation of the motion can be represented by a simple one d.o.f differential equation as long as the spring remains in contact with floor:

Figure 4.1.2 Model for a dropping body



$$m\ddot{x} + kx = mg \qquad \text{EQ. 4.1.5.1}$$

In this equation the damping effects are neglected to simplify the solution. The general solution of the differential equation is written as:

$$x = A\,Sin\omega t + B\,Cos\omega t + C \qquad \text{EQ. 4.1.5.2}$$

where the constants A, B and C are determined by the initial conditions:

$$\text{At t=0 =>}\ \ x = 0\ ,\ \dot{x} = \sqrt{2gh}\ ,\ \ddot{x} = g \qquad \text{EQ. 4.1.5.3}$$

ω is the natural frequency of the system: $\omega = \sqrt{\frac{k}{m}}$

Introducing these initial solutions into 4.1.5.3, the following result are obtained:

$$x = \frac{\sqrt{2gh}}{\omega} Sin\omega t + \frac{g}{\omega^2}\left(1 - Cos\omega t\right) \qquad \text{EQ. 4.1.5.4}$$

The same problem can be resolved by the numerical procedure explained in this chapter. Considering at first the following numerical values for the mass, the stiffness, the dropping height and the gravity:

$$m = 1\ ,\ \ k = 20\ \ ,\ \ \ \ h = 1\ \ ,\ \ \ \ g = 10 \qquad \text{EQ. 4.1.5.5}$$

From EQ. 4.1.5.1, the dynamic equilibrium equation or equation of motion is obtained as:

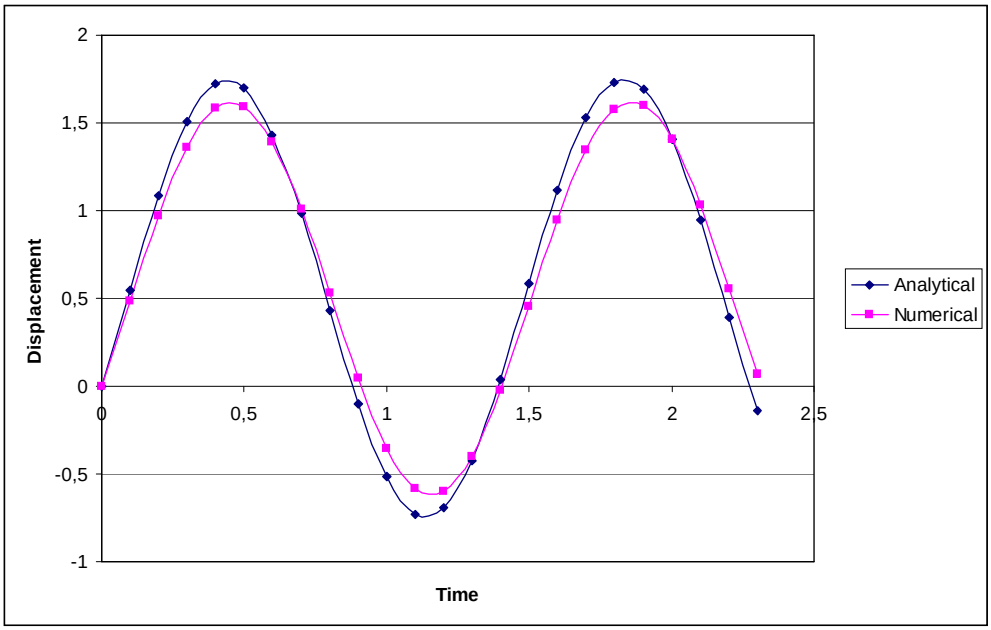
$$\ddot{x} + 20x = 10 \qquad \text{EQ. 4.1.5.6}$$

Using a step-by-step time discretization method with a central difference algorithm, for a given known step t_n the unknown kinematic variables for the next step are given by EQ. 4.1.5.6, EQ. 4.1.2.4 and EQ. 4.1.2.5:

$$\ddot{x}_n = 10 - 20x_n$$
$$\dot{x}_{n+1} = \ddot{x}_n \Delta t + \dot{x}_n \qquad \text{EQ. 4.1.5.7}$$
$$x_{n+1} = \dot{x}_n \Delta t + x_n$$

For the first time step the initial conditions are defined by EQ. 4.1.5.3. Using a constant time step $\Delta t = 0.1$ the mass motion can be computed. It is compared to the analytical solution given by EQ. 4.1.5.4 in Figure 4.1.3. The difference between the two results shows the time discretization error.

Figure 4.1.3 Obtained results for the example 4.1.3



4.1.6 Numerical Stability

The definition of numerically stability is similar to the stability of mechanical systems. *A numerical procedure is stable if small perturbations of initial data result in small changes in the numerical solution.* It is worthwhile to comment the difference between physical stability and numerical stability. Numerical instabilities arise from the discretization of the governing equations of the system, whereas physical instabilities are instabilities in the solutions of the governing equations independent of the numerical discretization. Usually numerical stability is only examined for physically stable cases. For this reason in the simulation of the physically unstable processes, it is not guaranteed to track accurately the numerical instabilities. Numerical stability of a physically unstable process cannot be examined by the definition given above. You establish the numerical stability criteria on the physically stable system and suppose that any stable algorithm for a stable system remains stable on an unstable system [36].

On the other hand, the numerical stability of time integrators discussed in the literature concerns generally linear systems and extrapolated to nonlinear cases by examining linearized models of nonlinear systems. The philosophy is the following: *if a numerical method is unstable for a linear system, it will be certainly unstable for nonlinear systems as linear cases are subsets of the nonlinear cases.* Therefore, the stability of numerical procedures for linear systems provides a useful guide to explore their behavior in a general nonlinear case.

To study the stability of the central difference time integration scheme, you establish the necessary conditions to ensure that the solution of equations is not amplified artificially during the step-by-step procedure. Stability also means that the errors due to round-off in the computer, do not grow in the integration. It is assured if the time step is small enough to accurately integrate the response in the highest frequency component.

4.1.7 Stability of the explicit scheme

In direct integration method, at time t_n the solutions for the prior steps are known and the solution for the time $t_{n+1} = t_n + \Delta t$ is required next. The equations to relate displacements, velocities and accelerations in a discrete time scale using the central difference time integration algorithm are given in Section 4.1.2. They can be rewritten as the following:

$$\dot{u}^{n+1} = \dot{u}^{n} + \frac{\Delta t}{2}\left(\ddot{u}^{n} + \ddot{u}^{n+1}\right) = \dot{u}^{n+\frac{1}{2}} + \frac{\Delta t}{2}\ddot{u}^{n+1}$$
$$u^{n+1} = u^{n} + \Delta t.\dot{u}^{n} + \frac{\Delta t^{2}}{2}.\ddot{u}^{n} = u^{n} + \Delta t.\dot{u}^{n+\frac{1}{2}} \qquad \text{EQ. 4.1.7.1}$$

For stability studies, aim to establish a recursive relationship to link the displacements at three consecutive time steps:

$$\begin{bmatrix} u^{n+1} \\ u^{n} \end{bmatrix} = [\mathrm{A}]\begin{bmatrix} u^{n} \\ u^{n-1} \end{bmatrix} + [\mathrm{L}] \qquad \text{EQ. 4.1.7.2}$$

where $[\mathrm{A}]$ is called amplification matrix. A spectral analysis of this matrix highlights the stability of the integration scheme:

The numerical algorithm is stable if and only if the radius spectral of $[\mathrm{A}]$ *is less than unity. In the other words when the module of all eigen values of* $[\mathrm{A}]$ *are smaller than unity the numerical stability is ensured.*

The stability of a numerical scheme can be studied using the general form of the 2x2 matrix $[\mathrm{A}]$:

$$[\mathbf{A}] = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} \\ \mathrm{A}_{21} & \mathrm{A}_{22} \end{bmatrix} \qquad \text{EQ. 4.1.7.3}$$

Then, the equations are developed for the systems with or without damping [55].
The eigen values of $[\mathrm{A}]$ are computed from the characteristic polynomial equation:

$$\det[\mathrm{A} - \lambda \mathrm{I}] = 0 \qquad \text{EQ. 4.1.7.4}$$
$$\lambda^2 - 2\mathrm{A}_1\lambda + \mathrm{A}_2 = 0$$

where $\mathrm{A}_1 = \frac{1}{2}\mathrm{tr}[\mathbf{A}] = \frac{1}{2}(\mathrm{A}_{11} + \mathrm{A}_{22})$,
$A_2 = \det[\mathrm{A}] = A_{11}A_{22} - A_{12}A_{21}$.

The eigen values are then obtained as:

$$\lambda_{1,2} = A_1 \pm \sqrt{A_1^2 - A_2} \qquad \text{EQ. 4.1.7.5}$$

If $\mathrm{A}_1^2 < \mathrm{A}_2$, eigen values are complex conjugate; if $\mathrm{A}_1^2 = \mathrm{A}_2$, they are real and identical; if $\mathrm{A}_1^2 > \mathrm{A}_2$, they are real and distinct. You intend to define a stability domain in the $(\mathrm{A}_1, \mathrm{A}_2)$-space, where the spectral radius $\rho([\mathbf{A}]) = \max(|\lambda_i(\mathrm{A}_1, \mathrm{A}_2)|) \leq 1$. The boundary of this domain is given by couples $(\mathrm{A}_1, \mathrm{A}_2)$ such as $\rho([\mathbf{A}]) = 1$. Three cases are to be considered:

1. Roots are real and one of them is equal to 1:

You then have:

$$1-2A_1+A_2=0 \qquad \text{EQ. 4.1.7.6}$$

This yields:

$$\begin{aligned} A_2 &= 2A_1-1 \\ \lambda_1 &= 1 \\ \lambda_2 &= A_2 \end{aligned} \qquad \text{EQ.4.1.7.7}$$

The corresponding part of the boundary of the stability domain is the segment analytically defined by $1-2A_1+A_2=0$ and $-1\le A_2\le 1$.

2. Roots are real and one of them is equal to –1 :

You then have:

$$1+2A_1+A_2=0 \qquad \text{EQ.4.1.7.8}$$

This yields:

$$\begin{aligned} A_2 &= -2A_1-1 \\ \lambda_1 &= -1 \\ \lambda_2 &= -A_2 \end{aligned} \qquad \text{EQ.4.1.7.9}$$

In this case, the corresponding part of the boundary is the segment given by $1+2A_1+A_2=0$ and $-1\le A_2\le 1$.

3. Roots are complex conjugate :

Their modulus is equal to 1. You then have, using $\lambda_{1,2}=e^{\pm i\alpha}$:

$$\begin{aligned} 0 &= e^{2i\alpha}-2A_1e^{i\alpha}+A_2 \\ &= (\cos 2\alpha-2A_1\cos\alpha+A_2)+i(\sin 2\alpha-2A_1\sin\alpha) \\ &= (2\cos\alpha(\cos\alpha-A_1)+A_2-1)+i(2\sin\alpha(\cos\alpha-A_1)) \end{aligned} \qquad \text{EQ.4.1.7.10}$$

This yields:

$$\begin{aligned} 2\cos\alpha(\cos\alpha-A_1)+A_2-1&=0 \\ 2\sin\alpha(\cos\alpha-A_1)&=0 \end{aligned} \qquad \text{EQ.4.1.7.11}$$

Since $\sin\alpha\neq 0$, you obtain:

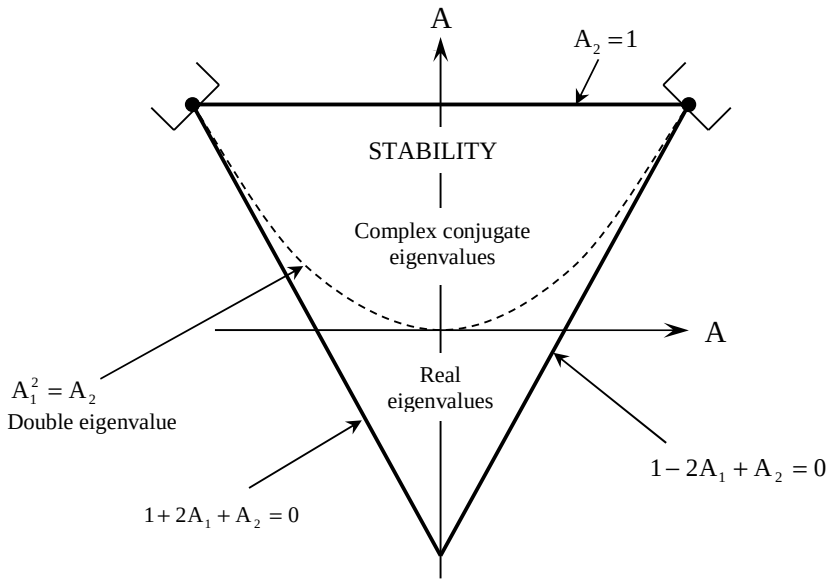
$$\begin{aligned} A_1 &= \cos\alpha \\ A_2 &= 1 \end{aligned} \qquad \text{EQ.4.1.7.12}$$

The corresponding part of the boundary is thus the segment given by $A_2=1$ and $-1\le A_1\le 1$.

The 3 segments introduced above define a closed contour. Point $A_1=A_2=0$ is located inside this contour and in this case, $\rho([\mathbf{A}])=0$. Since $\rho([\mathbf{A}])$ varies continuously with respect to A_1 and A_2, you can conclude that the stability domain corresponds to the interior of the contour. To precisely define the stability domain, you must

also have points leading to double eigen value of modulus 1, i.e. the intersections between the parabola $A_1^2 = A_2$ and the boundary of the domain. This corresponds to Points $(A_1, A_2) = (\pm 1, 1)$.

Figure 4.1.4 Stability domain



You can analytically summarize the description of the stability by means of the following two sets of conditions:

$$\begin{aligned} &(1) \quad -\frac{(A_2+1)}{2} \le A_1 \le \frac{(A_2+1)}{2}, \quad -1 \le A_2 < 1 \\ &(2) \quad -1 < A_1 < 1, \quad A_2 = 1 \end{aligned}$$

EQ.4.1.7.13

4.1.7.1 Numerical stability of undamped systems

The stability conditions developed in the previous section can be applied to a one degree-of-freedom of a system without damping. The dynamic equilibrium equation at time t_n is given by:

$$m\ddot{u}^n + ku^n = f^n$$

EQ.4.1.7.14

where *m* and *k* are respectively the nodal mass and stiffness. f^n is the external force at time t_n. Rewriting the central difference time integration equations from EQ. 4.1.7.1, you obtain:

$$\begin{aligned} u^{n+1} &= u^n + \Delta t.\dot{u}^{n+\frac{1}{2}} = u^n + \Delta t.\dot{u}^{n-\frac{1}{2}} + \Delta t^2.\ddot{u}^n \\ u^n &= u^{n-1} + \Delta t.\dot{u}^{n-\frac{1}{2}} \end{aligned}$$

EQ.4.1.7.15

and:

$$\ddot{u}^n = \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2}$$

EQ.4.1.7.16

Substituting these equations into EQ. 4.1.7.14, it yields:

$$m\frac{u^{n+1}-2u^{n}+u^{n-1}}{\Delta t^{2}}+ku^{n}=f^{n} \qquad \text{EQ.4.1.7.17}$$

This equation can be written as EQ. 4.1.7.2. Then the amplification matrix takes the expression:

$$[\mathrm{A}]=\begin{bmatrix} 2-\omega^{2}\Delta t^{2} & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{EQ.4.1.7.18}$$

where $\omega=\sqrt{\frac{k}{m}}$ is the angular frequency of the considered mode.

Comparing with EQ. 4.1.7.3, you have, $\mathrm{A}_1=1-\frac{\omega^{2}\Delta \mathrm{t}^{2}}{2}$ and $\mathrm{A}_2=1$. Stability is then given by:

$$-1<1-\frac{\omega^{2}\Delta \mathrm{t}^{2}}{2}<1 \qquad \text{EQ.4.1.7.19}$$

The right inequality is always true if $\omega \neq 0$. For, the particular case of $\omega=0$, the scheme is unstable. However, the analytical solution for a system with $\omega=0$ leads to an unbounded solution. The left inequality implies:

$$\boxed{\Delta t<\frac{2}{\omega}} \qquad \text{EQ.4.1.7.20}$$

4.1.7.2 Numerical stability with viscous damping - velocities at time steps

The dynamic equilibrium equation at time step *n* is written as:

$$m\ddot{u}^{n}+c\dot{u}^{n}+ku^{n}=f^{n} \qquad \text{EQ.4.1.7.21}$$

Using the equations:

$$\begin{aligned} \mathrm{u}^{\mathrm{n+1}} &= \mathrm{u}^{\mathrm{n}}+\Delta \mathrm{t}.\dot{\mathrm{u}}^{\mathrm{n}+\frac{1}{2}} \\ \mathrm{u}^{\mathrm{n}} &= \mathrm{u}^{\mathrm{n-1}}+\Delta \mathrm{t}.\dot{\mathrm{u}}^{\mathrm{n}-\frac{1}{2}} \end{aligned} \qquad \text{EQ.4.1.7.22}$$

Results in:

$$u^{n+1}-u^{n-1}=\Delta t\left(\dot{u}^{n+\frac{1}{2}}+\dot{u}^{n-\frac{1}{2}}\right) \qquad \text{EQ.4.1.7.23}$$

For the velocity, write the equations:

$$\dot{u}^{n} = \dot{u}^{n-\frac{1}{2}} + \frac{\Delta t}{2}\ddot{u}^{n}$$
$$\dot{u}^{n+\frac{1}{2}} = \dot{u}^{n} + \frac{\Delta t}{2}\ddot{u}^{n} \qquad \text{EQ.4.1.7.24}$$

to obtain:

$$\dot{u}^{n} = \frac{1}{2}\left(\dot{u}^{n-\frac{1}{2}} + \dot{u}^{n+\frac{1}{2}}\right) = \frac{1}{2\Delta t}\left(u^{n+1} - u^{n-1}\right) \qquad \text{EQ.4.1.7.25}$$

Substituting these equations into EQ.4.1.7.21, the recurring continuation equation on the displacement is written in the form:

$$m\frac{u^{n+1} - 2u^{n} + u^{n-1}}{\Delta t^{2}} + \frac{c}{2\Delta t}\left(u^{n+1} - u^{n-1}\right) + ku^{n} = f^{n} \qquad \text{EQ.4.1.7.26}$$

The equation can be rearranged to obtain the expression of the amplification matrix:

$$[\mathrm{A}] = \begin{bmatrix} \dfrac{2 - \omega^{2}\Delta t^{2}}{1 + \dfrac{c\Delta t}{2m}} & \dfrac{-1 + \dfrac{c\Delta t}{2m}}{1 + \dfrac{c\Delta t}{2m}} \\ 1 & 0 \end{bmatrix} \qquad \text{EQ.4.1.7.27}$$

This yields $A_1 = \dfrac{1 - \dfrac{\omega^{2}\Delta t^{2}}{2}}{1 + \dfrac{c\Delta t}{2m}}$ and $\mathrm{A}_2 = \dfrac{1 - \dfrac{\mathrm{c}\Delta \mathrm{t}}{2\mathrm{m}}}{1 + \dfrac{\mathrm{c}\Delta \mathrm{t}}{2\mathrm{m}}}$.

Stability is given by the set of conditions from EQ.4.1.7.13:

$$-\frac{1}{1 + \dfrac{c\Delta t}{2m}} \le \frac{1 - \dfrac{\omega^{2}\Delta t^{2}}{2}}{1 + \dfrac{c\Delta t}{2m}} \le \frac{1}{1 + \dfrac{c\Delta t}{2m}} \qquad \text{EQ.4.1.7.28}$$

$$-1 \le \frac{1 - \dfrac{c\Delta t}{2m}}{1 + \dfrac{c\Delta t}{2m}} < 1$$

The second expression is always verified for $c > 0$. It is the same for the right inequality of the first expression. The left inequality of the first expression leads to the condition on the time step:

$$\boxed{\Delta \mathrm{t} \le \frac{2}{\omega}} \qquad \text{EQ.4.1.7.29}$$

You find the same condition as in the undamped case, which echoes a conclusion given in [55]. You may yet remark that damping has changed the strict inequality into a large inequality, preventing from weak instability due to a double eigen value of modulus unity.

It is important to note that the relation EQ.4.1.7.29 is obtained by using the expression EQ.4.1.7.25 to compute nodal velocities at time steps. However, in an explicit scheme generally the mid-step velocities $\dot{u}^{n+\frac{1}{2}}$ and $\dot{u}^{n-\frac{1}{2}}$ are used. This will be studied in the next section.

4.1.7.3 Numerical stability with viscous damping - velocities at mid-steps

Considering the case in which damping effects cannot be neglected, you still would like to deal with decoupled equilibrium equations to be able to use essentially the same computational procedure. Except for the case of full modal projection which is a very expensive technique and practically unused, the damping matrix $[\mathbf{C}]$ is not diagonal, contrary to $[\mathbf{M}]$. The computation of the viscous forces with the exact velocity given by the integration algorithm requires the matrix $[\mathbf{M}]+\frac{\Delta t}{2}[\mathbf{C}]$ to be inverted, which can harm the numerical performances. You therefore often compute the viscous forces using the velocities at the preceding mid-step, which are explicit. This leads to an equilibrium at step *n* in the form:

$$m\ddot{u}^n + c.\dot{u}^{n-\frac{1}{2}} + ku^n = f^n \qquad \text{EQ.4.1.7.30}$$

The integration algorithm immediately yields:

$$\dot{u}^{n-\frac{1}{2}} = \frac{1}{\Delta t}\left(u^n - u^{n-1}\right) \qquad \text{EQ.4.1.7.31}$$

The recurring continuation becomes:

$$m\frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} + \frac{c}{\Delta t}\left(u^n - u^{n-1}\right) + ku^n = f^n \qquad \text{EQ.4.1.7.32}$$

As above, you obtain the amplification matrix:

$$[\mathrm{A}] = \begin{bmatrix} 2 - \omega^2\Delta t^2 - \dfrac{c\Delta t}{m} & -1 + \dfrac{c\Delta t}{m} \\ 1 & 0 \end{bmatrix} \qquad \text{EQ.4.1.7.33}$$

You have in this case $\mathrm{A}_1 = 1 - \dfrac{\omega^2\Delta t^2}{2} - \dfrac{c\Delta t}{2m}$ and $\mathrm{A}_2 = 1 - \dfrac{c\Delta t}{m}$.

Stability is again given by the set of conditions EQ.4.1.7.13:

$$-1+\frac{c\Delta t}{2m}\le 1-\frac{\omega^2\Delta t^2}{2}-\frac{c\Delta t}{2m}\le 1-\frac{c\Delta t}{2m} \qquad \text{EQ.4.1.7.34}$$

$$-1\le 1-\frac{c\Delta t}{m}<1$$

Right inequalities are always verified in both preceding expressions. Left inequalities now lead to two conditions on the time step:

$$\boxed{\Delta t\le \frac{-\frac{c}{m}+\sqrt{\frac{c^2}{m^2}+4\omega^2}}{\omega^2},\quad \Delta t\le\frac{2m}{c}} \qquad \text{EQ.4.1.7.35}$$

Therefore, the critical time step depends not only to ω but also to the mass and the damping. However, the critical time step depends only to ω when using the exact velocities to compute the viscous forces as described in the previous section.

4.1.7.4 Numerical stability with Rayleigh damping

The linearized equations of equilibrium governing the dynamic response of a finite element system can derived from the equations of motion given in Sections 3.5 and 3.6:

$$[M]\{\ddot{U}\}+[C]\{\dot{U}\}+[K]\{U\}=\{F\} \qquad \text{EQ.4.1.7.36}$$

In the case of direct step-by-step time integration, it is necessary to evaluate the damping matrix [C] explicitly. The Rayleigh damping method assumes that the matrix [C] is computed by the following equation:

$$[\mathrm{C}]=\alpha[\mathrm{M}]+\beta[\mathrm{K}] \qquad \text{EQ.4.1.7.37}$$

where $[\mathbf{C}]$ is the viscous damping matrix of the system,

$[\mathbf{M}]$ is the mass matrix,

$[\mathbf{K}]$ is the stiffness matrix.

As described in the preceding sections, the computation of the viscous forces by using velocities at time steps leads to obtain a non-diagonal matrix [C] which should be inverted in the resolution procedure. To avoid the high cost operations, generally the simplifications are made to obtain a diagonal matrix. Substituting the Rayleigh equation into EQ.4.1.7.36 and using the mid-step velocities for $\beta[K]$ terms and at step nodal velocities for $\alpha[\mathrm{M}]$ terms, the following expression is obtained:

$$[M]\{\ddot{u}^n\}+\alpha[M]\{\dot{u}^n\}+\beta[K]\left\{\dot{u}^{n-\frac{1}{2}}\right\}+[K]\{u^n\}=\{f^n\} \qquad \text{EQ.4.1.7.38}$$

Studying the equilibrium of a node to obtain a one dimensional equation of motion, write:

$$m\ddot{u}+c\dot{u}+ku=f \qquad \text{EQ.4.1.7.39}$$

where m is the modal mass,

c is the associated modal damping,

k is the nodal stiffness.

This leads to the following recurring continuation on the displacement:

$$m\frac{u^{n+1}-2u^n+u^{n-1}}{\Delta t^2}+\frac{\alpha m}{2\Delta t}\left(u^{n+1}-u^{n-1}\right)+\frac{\beta k}{\Delta t}\left(u^n-u^{n-1}\right)+ku^n=f^n \qquad \text{EQ.4.1.7.40}$$

The amplification matrix is then:

$$[\mathrm{A}]=\begin{bmatrix} \dfrac{2-\omega^2\Delta t^2-\beta\omega^2\Delta t}{1+\dfrac{\alpha\Delta t}{2}} & \dfrac{-1+\dfrac{\alpha\Delta t}{2}+\beta\omega^2\Delta t}{1+\dfrac{\alpha\Delta t}{2}} \\ 1 & 0 \end{bmatrix} \qquad \text{EQ.4.1.7.41}$$

In this case, $A_1=\dfrac{1-\dfrac{\omega^2\Delta t^2}{2}-\dfrac{\beta\omega^2\Delta t}{2}}{1+\dfrac{\alpha\Delta t}{2}}$ and $\mathrm{A}_2=\dfrac{1-\dfrac{\alpha\Delta t}{2}-\beta\omega^2\Delta t}{1+\dfrac{\alpha\Delta t}{2}}$.

Stability is obtained as before by means of the set of conditions from EQ.4.1.7.13:

$$\frac{-1+\dfrac{\beta\omega^2\Delta t}{2}}{1+\dfrac{\alpha\Delta t}{2}}\le\frac{1-\dfrac{\omega^2\Delta t^2}{2}-\dfrac{\beta\omega^2\Delta t}{2}}{1+\dfrac{\alpha\Delta t}{2}}\le\frac{1-\dfrac{\beta\omega^2\Delta t}{2}}{1+\dfrac{\alpha\Delta t}{2}} \qquad \text{EQ.4.1.7.42}$$

$$-1\le\frac{1-\dfrac{\alpha\Delta t}{2}-\beta\omega^2\Delta t}{1+\dfrac{\alpha\Delta t}{2}}<1$$

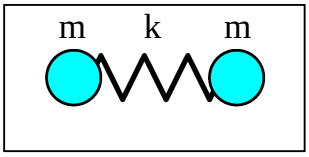
This again yields two conditions on the time step, coming from the left inequalities in both expressions:

$$\boxed{\Delta t\le\frac{-\beta\omega+\sqrt{\beta^2\omega^2+4}}{\omega},\quad \Delta t\le\frac{2}{\beta\omega^2}} \qquad \text{EQ.4.1.7.43}$$

It is equivalent to consider only the $\beta[\mathbf{K}]$ contribution in the damping for the computation of the time step, which appears to be logical since the $\alpha[\mathbf{M}]$ contribution is used with the exact velocity. It is advantageous to separate the two contributions, restrictions of the time step then becoming lighter. It can be shown that for the complete treatment of the Rayleigh damping using mid-step velocities, the stability conditions can be given by the following expressions:

$$\Delta t\le\frac{-\alpha-\beta\omega^2+\sqrt{\left(\alpha+\beta\omega^2\right)^2+4\omega^2}}{\omega},\quad \Delta t\le\frac{2}{\alpha+\beta\omega^2} \qquad \text{EQ.4.1.7.44}$$

4.1.7.5 Example: Critical time step for a mass-spring system



Consider a free mass-spring system without damping.
The governing differential equation can be written as:

$$M\ddot{X} + KX = 0 \qquad \text{(a)}$$

The element matrix expressions are given as:

$$[M] = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \qquad [K] = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The general solution is assumed in the form of:

$$X = Cos(\omega t - \alpha)[I]\Psi \qquad \text{(b)}$$

where the angular frequency ω and the phase angle α are common for all X_i. α and Ψ_i are the constants of integration to be determined from the initial conditions of the motion and ω is a characteristic value (eigen value) of the system. Substituting (b) into (a) yields:

$$\left(K - \omega^2 M\right)\Psi Cos(\omega t - \alpha) = 0 \qquad \text{(c)}$$

The consistency of (c) for $\Psi Cos(\omega t - \alpha) \neq 0$ requires that:

$$\det\left(K - \omega^2 M\right) = 0 \quad => \quad \boxed{\omega^2 = \frac{2k}{m}} \qquad \text{(d)}$$

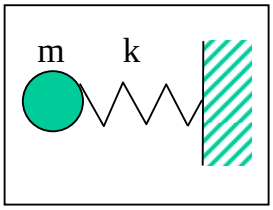
Assuming the following numerical values m = 1 and k =10, you have $\omega = \sqrt{\frac{2k}{m}} = 4.472136$. The critical time step of the system is given by EQ. 4.1.7.20:

$$\Delta t \le \frac{2}{\omega} \quad \Rightarrow \quad \Delta t \le \frac{2}{4.472136} \quad \Rightarrow \quad \Delta t \le 0.4472$$

4.1.7.6 Example: Critical time step for dropping body example

A dropping body is studied in Example 4.1.5 with analytical and numerical approaches. As shown in Figure 4.1.3 the numerical results are closed to the analytical solution if you use a step-by-step time discretization method with a constant time step Δt =0.1. This value is less than the critical time step obtained by:

$$\Delta t_{cr} = \frac{2}{\omega_{\max}}$$

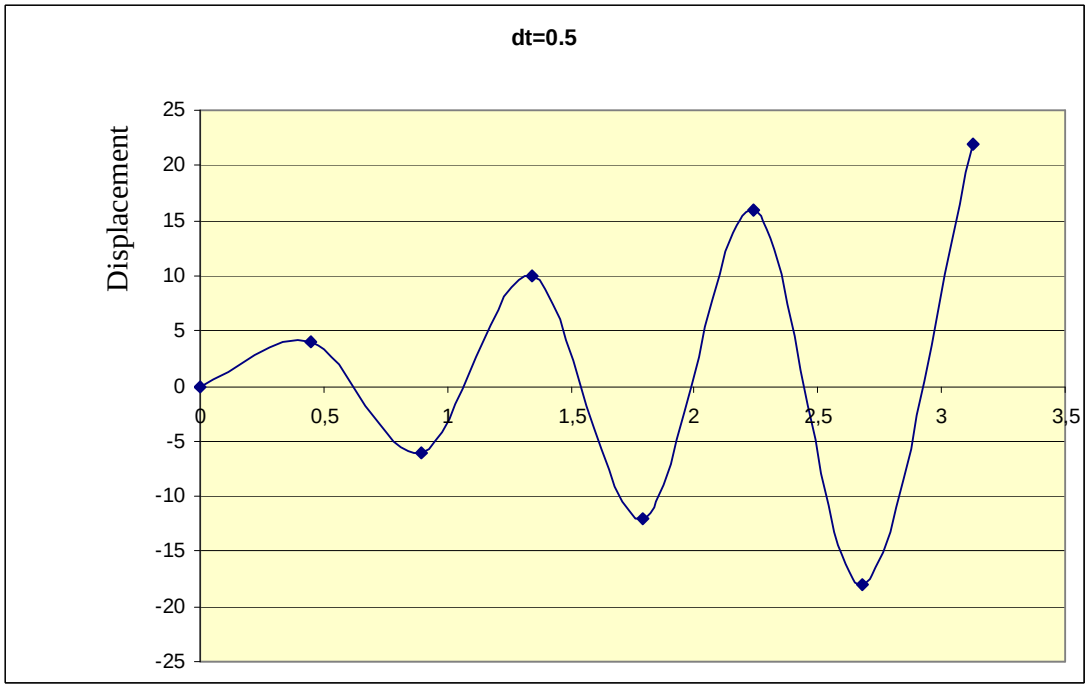


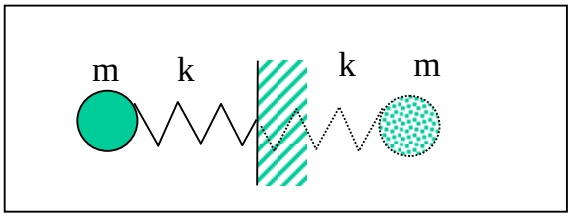
which may be computed as:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{20} \Rightarrow \quad \Delta t_{cr} = \frac{2}{4.472136} \quad \Rightarrow \quad \Delta t_{cr} = 0.4472$$

Therefore, the used time step in the Example 4.1.5 ensures the stability of the numerical scheme as it is less than the critical value. Now using the values higher than or equal to the critical time step, the solution diverges towards the infinity as shown in Figure 4.1.5.

Figure 4.1.5 Numerical instability for Example 4.1.7.6 using over critical time steps





It is worthwhile to comment that in a general explicit finite element program as *RADIOSS*, the critical time step is computed for an entire element (two nodal masses and stiffness for spring element). In case of dropping body example, the mass-spring system can be compared by analogy to a two-node mass-spring system where the global stiffness is twice softer. The critical time step is then computed using the nodal time step of the entire element (refer to the following sections for more details on the computation of nodal time step).

4.1.8 Stability by Courant condition

RADIOSS uses elements with a lumped mass approach. This reduces computational time considerably as no matrix inversion is necessary to compute accelerations.

The integration scheme used by RADIOSS is based on the central difference integration scheme which is conditionally stable, i.e. the time step must be small enough to assure that the solution does not grow boundlessly.

The stability condition is given in the last sections. For a system without damping, it can be written in a closed form:

$$\Delta t \le \frac{2}{\omega_{max}} \qquad \text{EQ. 4.1.8.1}$$

where ω_{max} is the highest angular frequency in the system:

$$\det\left(K - \omega^2 M\right) = 0 \qquad \text{EQ. 4.1.8.2}$$

where *K* and *M* are respectively the stiffness and the mass matrices of the system.

The time step restriction given by EQ. 4.1.8.1 was derived considering a linear system (see Section 4.1.7), but the result is also applicable to nonlinear analysis since on a given step the resolution is linear. However, in nonlinear analysis the stiffness properties change during the response calculation. These changes in the material and the geometry influence the value of ω_{max} and in this way the critical value of the time step.

The above point can be easily pointed out by using a nonlinear spring with increasing stiffness in Example 4.1.5. It can be shown that the critical time step decreases when the spring becomes stiffer. Therefore, if a constant time step close to the initial critical value is considered, a significant solution error is accumulated over steps when the explicit central difference method is used.

Another consideration in the time integration stability concerns the type of problem which is analyzed. For example in the analysis of wave propagation, a large number of frequencies are excited in the system. That is not always the case of structural dynamic problems. In a wave shock propagation problem, the time step must be small enough in order to excite all frequencies in the finite element mesh. This requires short time step so that the shock wave does not miss any node when traveling through the mesh. It follows that the time step should be limited by the following relation:

$$\boxed{\Delta t \le \frac{l_c}{c}} \qquad \text{EQ.4.1.8.3}$$

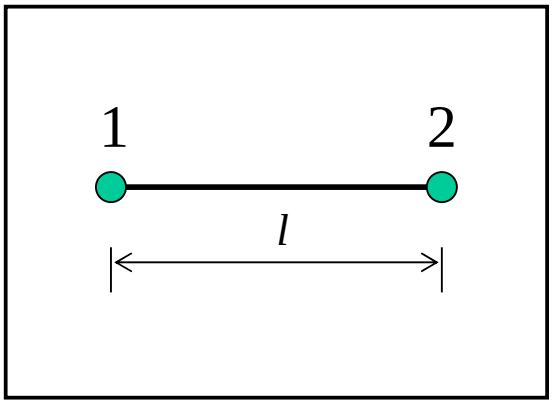
where:

l_c is the characteristic element length, representing the shortest road for a wave arriving on a node to cross the element,

c is the speed of sound in the material, Δt is the time step.

The condition EQ. 4.1.8.3 gives a severe time step restriction with respect to stability time step i.e. $\Delta t < \frac{2}{\omega}$.
It can easily be shown that for a simple case of a bar element, the two expressions EQ. 4.1.8.3 and EQ. 4.1.8.1 are equivalent.

Figure 4.1.6 Bar element



If a uniform linear-displacement bar element is considered, (Figure 4.1.7), and a lumped mass formulation at the nodes is used, the highest frequency of this element can be obtained by a resolution of an eigen value problem:

$$[K]\{X\} = \omega^2 [M]\{X\} \qquad \text{EQ. 4.1.8.4}$$

$$\det\left([K] - \omega^2 [M]\right) = 0 \qquad \text{EQ. 4.1.8.5}$$

For a lumped mass bar, you have:

$$[K] = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \text{EQ. 4.1.8.6}$$

$$[M] = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ EQ. 4.1.8.7

where *m* and *k* are respectively the nodal mass and stiffness of the bar:

$$k = \frac{EA}{l} \; ; \; m = \frac{A\rho l}{2}$$ EQ. 4.1.8.8

Equation 4.1.8.5 yields:

$$\left(k - m\omega^2\right)^2 - k^2 = 0$$ EQ. 4.1.8.9

then:

$$\omega = \sqrt{\frac{2k}{m}}$$ EQ. 4.1.8.10

which can be simplified with EQ. 4.1.8.8 to obtain:

$$\omega = \frac{2}{l}\sqrt{\frac{E}{\rho}} = \frac{2c}{l}$$ EQ.4.1.8.11

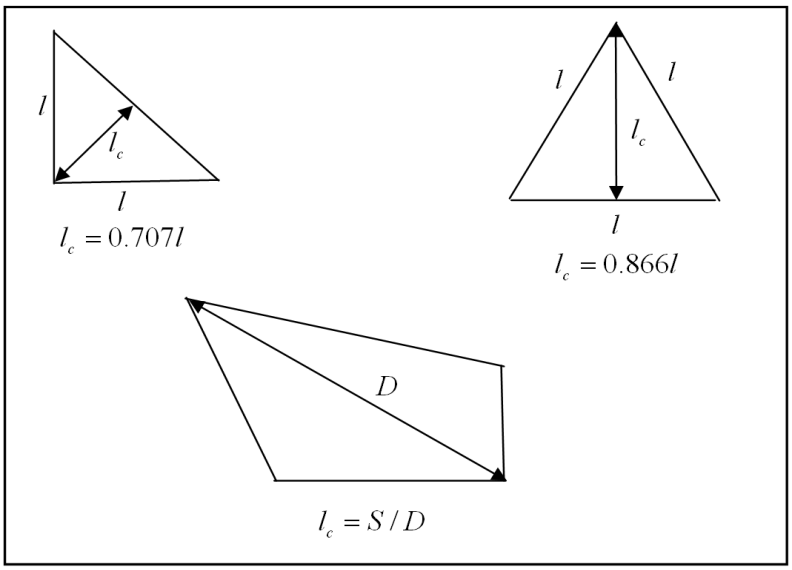
where *c* is the speed of sound in the material and its expression is given as:

$$c = \sqrt{\frac{E}{\rho}}$$ EQ. 4.1.8.12

with ρ the material density and *E* the Young's modulus. Combining EQ. 4.1.8.11 and EQ. 4.1.8.1, you obtain:

$$\Delta t \le \frac{l}{c}$$ EQ. 4.1.8.13

Figure 4.1.7 Element Characteristic Lengths



This relation is that of EQ. 4.1.8.3 and shows that the critical time step value in the explicit time integration of dynamic equation of motion can be carried out by the interpretation of shock wave propagation in the material. This is shown for the first time by Courant *et al.* in 1928 [56]. In spite of their works are limited to simple cases, the same procedure can be used for different kinds of finite elements. The characteristic lengths of the elements are found and EQ. 4.1.8.3 is written for all elements to find the most critical time step over a mesh. Regarding to the type (shape) of element, the expression of characteristic length is different. Figure 4.1.7 shows some typical cases for elements with one integration point.

4.1.9 Time Step Control in RADIOSS

The time incrementation in RADIOSS is fully automatic and *a priori* requires no user intervention. The step used for time integration (or moving forward in time) can be calculated using two different methods. The method used depends on the type of simulation being performed.

The two time step methods are:

- the element time step,
- the nodal time step.

The time step used by the solver is the largest possible time step, as determined by the Courant condition that will maintain stability. If the default large strain formulation is used, the time step is computed at each cycle. Large element deformation can give a large time step decrease. If the deformation is too large, negative volumes can result, which make it impossible to invert the Jacobian matrix and to integrate the stress over the volume. If the small strain formulation is used, assuming a constant Jacobian matrix during time and also a constant volume, all spatial variables are defined at $t = 0$. This is either the beginning of the analysis or the time at which the small strain formulation is initiated. If the sound speed is constant, the time step thus becomes constant. Using this formulation, the time step has no effect on the computation since the initial volume is used.

4.1.9.1 Element time step control

The stable element time step was detailed in Section 4.1.8 and is restated as:

$$\Delta t = \min_{Elements}\left(\frac{l}{c}\right) \qquad \text{EQ. 4.1.9.1}$$

where:

- l is the element characteristic length,
- c is the speed of sound in the material.

This is the default setting.

The element time step is computed at the same time as the internal forces. The characteristic length and the sound speed are computed for each element in every cycle.

The computed time step is compared to a minimum time step value and a scale factor is applied to ensure a conservative bound. Different minimum time step values can be given to different element types by using the option: */DT/Keyword* where *Keyword* is defined in the user manual as the element type.

If deformation is large enough for the time step to reach the minimum defined value, three options are possible under the user's control:

- Stop the analysis when the minimum time step value is reached. This is the default for brick and quadrilateral elements.
- Delete the element(s) defining the time step. This is the default for shell elements.
- Implement small strain formulation using a constant time step. This only works for shell and brick elements.

These options are defined using a third keyword: *STOP*, *DEL*, *CST*, *AMS* or *SET*.

4.1.9.2 Nodal time step control

The nodal time step is calculated after the computation of all the internal forces at each node using the following equation:

$$\Delta t = \min_{Nodes} \sqrt{\frac{2m}{k}} \qquad \text{EQ. 4.1.9.2}$$

where *m* is the nodal mass and k an equivalent nodal stiffness.

The nodal stiffness is one half of eigen value from element stiffness matrix; for a truss element this value is equal to the diagonal term of the stiffness matrix. It is computed from the accumulation of element and interface stiffness'. These stiffness' are obtained during internal force computation.

For a regular mesh, the element time step and nodal time step conditions are identical. For example, take a truss element, Figure 4.5.2. Using an element time step condition:

$$\Delta t_{element} = \frac{l}{c} = \frac{l}{\sqrt{\frac{E}{\rho}}} \qquad \text{EQ. 4.1.9.3}$$

The nodal time step condition is written as:

$$\Delta t_{nodal} = \sqrt{\frac{2m}{k}} \qquad \text{EQ. 4.1.9.4}$$

with $m = \frac{1}{2}\rho A l$ and $k = \frac{EA}{l}$ EQ. 4.1.9.5

Therefore:

$$\Delta t_{nodal} = \sqrt{\frac{\rho A l}{EA/l}} = \frac{l}{\sqrt{\frac{E}{\rho}}} = \frac{l}{c} = \Delta t_{element} \qquad \text{EQ. 4.1.9.6}$$

To select the nodal time step when running RADIOSS the option /DT/NODA has to be used.

As for the element time step, minimum time step and scale factors are required. The default value for the scale factor is 0.9. If the minimum time step is reached, the analysis can either be stopped or a mass scaling formulation can be applied. In the latter case, mass is added to the affected nodes so that the time step remains constant at the minimum value. This option can be enabled using the same third keyword as used in the element time step control. It must be checked that added masses do not affect the accuracy of results. If one uses the nodal time step, the element time step is ignored.

4.1.9.3 Interface time step control

Finally, the time step is influenced by existence of interfaces. The interface time step control depends on the type of interface being used.

For the interfaces in which the contact conditions are defined by applying kinematic conditions, no time step restriction is required. This is the case of interface type 2 of RADIOSS.

In addition, for the interfaces types 3, 4, 5, and 8 in RADIOSS a small stiffness is used. Therefore, the interfaces are stable if a time step scale factor less than or equal to 0.9 is used.

Types 7, 10 and 11 interfaces use a variable stiffness and if this stiffness is not small compared to the element stiffness, a stability condition must be adhered to.

For interfaces 3, 4, 5, 7, 8, and 10, there are three possibilities that can be selected shown in Table 4.1.1.

Table 4.1.1 - Interface Time Step Cases

Default (element) time step without interfaces types 7, 10 or 11	Element time step is computed and a scaling factor of 0.9 (default) is applied.
Option /DT/NODA is used with or without interface types 7 and/or 10, 11	Nodal time step is computed and a scaling factor of 0.9 (default) is applied.
Default time step with interface types 7 or 10, 11	Nodal and element time steps are computed and the smallest is used.

If the deletion option is applied with the */DT/INTER/DEL* interface time step control, the node controlling the minimum time step is deleted from the interface.

Mass scaling, where mass is added to an interface node, can be enabled using the option: /*DT/INTER/CST*

4.1.10 Time step control limitations

Many of the time step control options influence the solution results. The solution of the nonlinear dynamic response of a finite element system accurate is the numerical model correctly represents the physical model. The critical time step given for finite element system is determined by a theoretical approach in which the highest frequency of the discretized system controls this value. Therefore, the time step limitations are related to the model and cannot be changed without incidence on the quality of results.

Using the "DEL" option can significantly alter the model, since elements and nodes are removed without replacement. In fact, mass and/or volume is lost. Using either /DT/NODA/CST or /DT/INTER/CST will add mass to the model to allow mathematical solution. The added mass will increase the kinetic energy. This should be checked by the user to see if there is a significant effect. Switching to a small strain option using brick or shell elements also introduces errors as it was seen in Section 2.12.

Generally, in the study of the nonlinear dynamic response of a system, three physical laws have to be respected:

- Conservation of mass,
- Conservation of energy,
- Conservation of momentum dynamic equilibrium.

The last one is generally respected as the equation of motion is resolved at each resolution cycle. However, in the case of adding masses especially when using /DT/NODA/CST option, it is useful to verify the momentum variation. The two other conservation laws are not explicitly satisfied. They should be checked a posteriori after computation to ensure the validity of the numerical model with respect to the physical problem.

4.1.10.1 Time step example

Explicit scheme stability condition

$$M\ddot{X} + KX = 0 \qquad \text{EQ. 4.1.10.1}$$

$$\ddot{X} = M^{-1}KX \qquad \text{EQ. 4.1.10.2}$$

$$\dot{X}_{n+\frac{1}{2}} = \dot{X}_{n-\frac{1}{2}} + dt\ddot{X}_n \qquad \text{EQ. 4.1.10.3}$$

$$X_n = X_{n-1} + dt\dot{X}_{n-\frac{1}{2}} \qquad \text{EQ. 4.1.10.4}$$

EQ. 4.1.10.2 and *n* EQ. 4.1.10.3 are added:

$$\dot{X}_{n+\frac{1}{2}} = \dot{X}_{n-\frac{1}{2}} + dtM^{-1}KX_n \qquad \text{EQ. 4.1.10.5}$$

The following equation is obtained:

$$\dot{X}_{n-\frac{1}{2}} = \dot{X}_{n-\frac{1}{2}-1} + dtM^{-1}KX_{n-1} \qquad \text{EQ. 4.1.10.6}$$

When EQ. 4.1.10.6 is subtracted from EQ. 4.1.10.5, and EQ. 4.1.10.4 is used:

$$\dot{X}_{n+\frac{1}{2}} = A\dot{X}_{n-\frac{1}{2}} - X_{n-\frac{1}{2}-1} \qquad \text{EQ. 4.1.10.7}$$

Where $A = 2I + dt^2 M^{-1}K$

For non divergence of EQ. 4.1.10.7:

=> λ : largest eigen value of A-I is smaller than 1

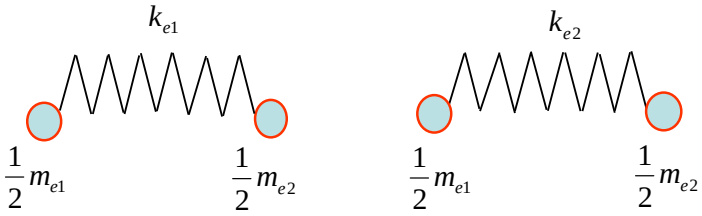
=> $\det\left(\lambda I + I + dt^2 M^{-1}K\right) = 0$

=> $\det\left(I\left((\lambda+1)/dt^2\right) + M^{-1}K\right) = 0$

=> $2dt^2 >$ largest eigen value of $M^{-1}K$

=> $2dt^2 <$ smallest eigen value of $2K^{-1}M$

Application



$$dt_1 = \sqrt{\frac{2m_{el}}{k_{el}}}$$

$$dt_2 = \sqrt{\frac{2m_{e2}}{k_{e2}}}$$

$$dt_e = \min\left(dt_1, dt_2\right)$$

$$dt_n = \sqrt{\frac{m_{e1}+m_{e2}}{k_{el}+k_{e2}}}$$

Interface

Interfaces have stiffness but no mass: $dt_e = 0$

Solution 1: (interface type 3, 4, 5)

$k_{interface} << k_{elements}$

Solution 2: nodal time step (interface type 7 and type 10):

$$dt_n = \sqrt{\frac{2m}{k}} \quad k = \sum k_{\text{interface}} + \sum k_{\text{elements}}$$

Kinematic time step (interface type 7):

dt < (gap-pene)/2 V

V: impact speed

4.2 Large Scale Eigen value Computation

The numerical solution of large scale algebraic eigen value problems is now available thanks to new methods and software. A class of methods called Krylov subspace projection methods is used. The well known Lanczos method is the first one. The Arnoldi method is a generalization of Lanczos method applied to the non-symmetric case. A variant of Arnoldi-Lonczos scheme called the Implicitly Restarted Arnoldi Method [97] is a part of public domain software package called ARPACK which is integrated in RADIOSS. Restarting is introduced as a way to overcome intractable storage and computational requirements in the original Arnoldi method. Implicit restarting is a variant of restarting which may be considered as a truncated form of the powerful implicitly shifted QR technique that is suitable for large scale problems. It provides a mean to approximate a few eigen values with user specified properties in space proportional to the number of eigen values required. The details of the method are not explained here. The reader is invited to consult [97] for a deep lecture.

4.3 Combining Modal Reduction on sub-domain and Direct Integration Methods

A domain decomposition method allowing the combination of nonlinear sub-domains with linear modal sub-domains has been proposed in [107]. With this technique, the displacement field in the linear sub-domains is projected on a local basis of reduction modes calculated on the detailed geometry and the kinematic continuity relations are written at the interface in order to recombine the physical kinematic quantities of reduced sub-domains locally. The method yields promising save of computing time in industrial applications. However, the use of modal projection is limited to linear sub-domains. In the case of overall rigid-body motion with small local vibrations, the geometrical nonlinearity of sub-domains must be taken into account. Therefore, the projection cannot be used directly even thought the global displacements may still be described by a small number of unknowns; for example six variables to express motion of local frame plus a set of modal coordinates in this frame. This approach is used in the case of implicit framework in [108]. In the case of direct integration with an explicit scheme an efficient approach is presented in [109]. One of the main problems is to determine the stability conditions for the explicit integration scheme when the classical rotation parameters as Euler angles or spin vectors are used. A new set of parameters, based on the so-called ‘frame-mass’ concept is introduced to describe the global rigid body motion. The position and the orientation of the local frame are given by four points where the distances between the points are kept constant during the motion. In this way, only the displacement type d.o.f. is dealt and the equations of motion are derived to satisfy perfectly the stability conditions. This approach, which was integrated in RADIOSS V5, will be presented briefly here. The reader is invited to consult [109] for a detailed view.

4.3.1 Linear modal reduction

A modal reduction basis is defined on one or more sub-domains of the decomposition. The definition of this basis is completely arbitrary. Any combination of eigen modes and static corrections can be used. All these modes are orthogonalized with respect to the finite element mass matrix in order for the projected mass matrix to be diagonal and suitable for an explicit solver.

Considering the case of a structure divided into two sub-domains, assume that modal reduction is used for linear Sub-domain 1. Thus, the displacement field of this sub-domain is projected onto the reduction vectors:

$$U_1(t) = \sum_1^n \alpha_{1i}(t)\Phi_{1i} = \Phi_1\alpha_1(t) \qquad \text{EQ. 4.3.1.1}$$

with U_1 vector of discretized displacements in Sub-domain 1, α_1 vector of modal participations. This naturally yields:

$$\begin{aligned} \dot{U}_1 &= \Phi_1\dot{\alpha}_1 \\ \ddot{U}_1 &= \Phi_1\ddot{\alpha}_1 \end{aligned} \qquad \text{EQ. 4.3.1.2}$$

The number of modal unknowns α_1 chosen is much smaller than the original number of degrees of freedom of Sub-domain 1.

In order to obtain the new coupled system, the dynamic equilibrium of sub-domain 1 must be projected onto the reduction basis and the velocities involved in the kinematic relations must be expressed in terms of the modal coordinates. Thus, write the new matrix system for a single time scale as:

$$\begin{bmatrix} \frac{\Delta t}{2}\hat{M}_1 & 0 & -\frac{\Delta t}{2}\hat{C}_1{}^T \\ 0 & \frac{\Delta t}{2}M_2 & -\frac{\Delta t}{2}C_2{}^T \\ -\frac{\Delta t}{2}\hat{C}_1 & -\frac{\Delta t}{2}C_2 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\alpha}_1^{n+1} \\ \ddot{U}_2^{n+1} \\ \Lambda \end{bmatrix} = \begin{bmatrix} \frac{\Delta t}{2}\hat{F}_1^{n+1} \\ \frac{\Delta t}{2}F_2^{n+1} \\ \hat{C}_1{}^P\dot{\alpha}_1^{n+1} + C_2{}^P\dot{U}_2^{n+1} \end{bmatrix} \qquad \text{EQ. 4.3.1.3}$$

where

$$\begin{aligned} \hat{M}_1 &= \Phi_1{}^T M_1 \Phi_1 \\ \hat{F}_1 &= \Phi_1{}^T F_{ext1} - \hat{K}_1\alpha_1 \\ \hat{K}_1 &= \Phi_1{}^T K_1 \Phi_1 \\ \hat{C}_1 &= C_1\Phi_1 \end{aligned}$$

The structure of this system is strictly identical to that which existed before reduction. Therefore, use exactly the same resolution process and apply the multi-time-step algorithm.

The time step for a reduced sub-domain is deduced from the highest eigen frequency of the projected system in order to preserve the stability of the explicit time integration. This time step is often larger than that given by the Courant condition with the finite element model before reduction.

4.3.2 Modal reduction with finite overall rotations

Since large rotations are highly nonlinear [50], the displacement field in a sub-domain undergoing finite rotations cannot be expressed as a linear combination of constant modes. However, the rigid contribution to the displacement field creates no strain. In the case of small strains and linear behavior, the local vibrating system can still be projected onto a basis of local reduction modes. Then, take into account the rotation matrix from the initial global coordinate system to the local coordinate system and its time derivatives. A classical parameterization of this rotation, for example, Euler angles, would introduce nonlinear terms involving

velocities. Since these quantities, in the central difference scheme, are implicit, this would require internal iterations in order to solve the equilibrium problem, a situation you clearly want to reduce the computation time due to the reduction.

Classically, the displacement field of a rotating and vibrating sub-domain is decomposed into a finite rigid-body contribution and a small-amplitude vibratory contribution measured in a local frame. The large rigid motion is represented using the so-called "four-mass" approach [109].

Four points (O, A, B, C) in space are arbitrarily chosen to represent the position of a local frame attached to the sub-domain. In order to simplify the local equations, choose these points so that they constitute an ortho-normal frame.

Note: The four points defining the local frame do not have to coincide with nodes of the mesh.

4.3.2.1 Decomposition of displacement field

The global displacements of these four points are the unknowns describing the rigid motion of the sub-domain. These are classical displacement-type parameters. The relative distances between these points are kept constant during the dynamic calculation through external links. This enables us to express the total continuous displacement field of the sub-domain as follows:

$$\mathrm{u} = X\mathrm{u}_A + Y\mathrm{u}_B + Z\mathrm{u}_C + (1 - X - Y - Z)\mathrm{u}_O + \mathrm{P}\mathrm{u}_L = \mathrm{u}_E + \mathrm{P}\mathrm{u}_L \qquad \text{EQ. 4.3.2.1}$$

where X, Y, Z are the coordinates in the local frame $[O, A, B, C]$,

P is the rotation matrix expressing the transformation from the local to the global coordinates: since $\left(\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}\right)$ are unit vectors, $\mathrm{P} = \left[\overrightarrow{OA}; \overrightarrow{OB}; \overrightarrow{OC}\right]$.

In order to express the dynamic equilibrium, EQ. 4.3.1.4 must be derived with respect to time to yield velocities and accelerations.

$$\begin{aligned} \dot{\mathrm{u}} &= X\dot{\mathrm{u}}_A + Y\dot{\mathrm{u}}_B + Z\dot{\mathrm{u}}_C + (1 - X - Y - Z)\dot{\mathrm{u}}_O + \dot{\mathrm{P}}\mathrm{u}_L + \mathrm{P}\dot{\mathrm{u}}_L \\ \ddot{\mathrm{u}} &= X\ddot{\mathrm{u}}_A + Y\ddot{\mathrm{u}}_B + Z\ddot{\mathrm{u}}_C + (1 - X - Y - Z)\ddot{\mathrm{u}}_O + \ddot{\mathrm{P}}\mathrm{u}_L + \mathrm{P}\ddot{\mathrm{u}}_L + 2\dot{\mathrm{P}}\dot{\mathrm{u}}_L \end{aligned} \qquad \text{EQ. 4.3.2.2}$$

The time derivatives of the rotation matrix are given by:

$$\begin{aligned} \dot{\mathrm{P}} &= [\dot{\mathrm{u}}_A - \dot{\mathrm{u}}_O; \dot{\mathrm{u}}_B - \dot{\mathrm{u}}_O; \dot{\mathrm{u}}_C - \dot{\mathrm{u}}_O] \\ \ddot{\mathrm{P}} &= [\ddot{\mathrm{u}}_A - \ddot{\mathrm{u}}_O; \ddot{\mathrm{u}}_B - \ddot{\mathrm{u}}_O; \ddot{\mathrm{u}}_C - \ddot{\mathrm{u}}_O] \end{aligned} \qquad \text{EQ. 4.3.2.3}$$

Thus, the fields in EQ. 4.3.2.2 have the following expression:

$$\begin{aligned} \dot{\mathrm{u}} &= \left(X + \mathrm{u}_L{}^X\right)\dot{\mathrm{u}}_A + \left(Y + \mathrm{u}_L{}^Y\right)\dot{\mathrm{u}}_B + \left(Z + \mathrm{u}_L{}^Z\right)\dot{\mathrm{u}}_C + \left[1 - \left(X + \mathrm{u}_L{}^X\right) - \left(Y + \mathrm{u}_L{}^Y\right) - \left(Z + \mathrm{u}_L{}^Z\right)\right]\dot{\mathrm{u}}_O \\ &\quad + \mathrm{P}\dot{\mathrm{u}}_L \\ \ddot{\mathrm{u}} &= \left(X + \mathrm{u}_L{}^X\right)\ddot{\mathrm{u}}_A + \left(Y + \mathrm{u}_L{}^Y\right)\ddot{\mathrm{u}}_B + \left(Z + \mathrm{u}_L{}^Z\right)\ddot{\mathrm{u}}_C + \left[1 - \left(X + \mathrm{u}_L{}^X\right) - \left(Y + \mathrm{u}_L{}^Y\right) - \left(Z + \mathrm{u}_L{}^Z\right)\right]\ddot{\mathrm{u}}_O \\ &\quad + \mathrm{P}\ddot{\mathrm{u}}_L + 2\dot{\mathrm{P}}\dot{\mathrm{u}}_L \end{aligned} \qquad \text{EQ. 4.3.2.4}$$

where $\mathrm{u}_L{}^X, \mathrm{u}_L{}^Y, \mathrm{u}_L{}^Z$ are the components of the local displacement in the local frame. The assumption of small perturbations in the local frame enables us to consider that the rigid and the deformed configurations are the same, that is, you can neglect $\mathrm{u}_L{}^X, \mathrm{u}_L{}^Y, \mathrm{u}_L{}^Z$ compared to the local coordinates, X, Y, Z . Thus, you get simplified expressions of the velocity and acceleration fields:

$$\dot{u} = X\dot{u}_A + Y\dot{u}_B + Z\dot{u}_C + (1 - X - Y - Z)\dot{u}_O + \dot{P}u_L$$
$$\ddot{u} = X\ddot{u}_A + Y\ddot{u}_B + Z\ddot{u}_C + (1 - X - Y - Z)\ddot{u}_O + P\ddot{u}_L + 2\dot{P}\dot{u}_L \quad \text{EQ. 4.3.2.5}$$

To express the weak form of the dynamic equilibrium, you also need the variation δu of the displacement field:

$$\delta \mathbf{u} = X\delta \mathbf{u}_A + Y\delta \mathbf{u}_B + Z\delta \mathbf{u}_C + (1 - X - Y - Z)\delta \mathbf{u}_O + \delta \mathbf{P}\mathbf{u}_L + \mathbf{P}\delta \mathbf{u}_L \quad \text{EQ. 4.3.2.6}$$

where $\delta \mathbf{P} = [\delta \mathbf{u}_A - \delta \mathbf{u}_O ; \delta \mathbf{u}_B - \delta \mathbf{u}_O ; \delta \mathbf{u}_C - \delta \mathbf{u}_O]$

Again, the same assumption as above allows us to simplify this expression:

$$\delta \mathbf{u} = X\delta \mathbf{u}_A + Y\delta \mathbf{u}_B + Z\delta \mathbf{u}_C + (1 - X - Y - Z)\delta \mathbf{u}_O + \mathbf{P}\delta \mathbf{u}_L \quad \text{EQ. 4.3.2.7}$$

4.3.2.2 Local reduced dynamic system

The local dynamic equilibrium of the sub-domain is given by:

$$\rho \ddot{u} - f_{int}(u) = 0 \quad \text{EQ.4.3.2.8}$$

The principle of virtual work yields a weak form of this equilibrium, taking into account, Dirichlet-type boundary conditions:

$$\int_\Omega \rho \delta u^T \ddot{u} d\Omega - \int_\Omega \delta u^T f_{int}(u) d\Omega = 0 \quad \text{EQ.4.3.2.9}$$

$\forall \delta u$ verifies the kinematic boundary conditions

Ω is the volume of the sub-domain.

To introduce EQ. 4.3.1.4 into this weak form of the equilibrium, you must express with the new parameterization the virtual works of both the internal and external forces and the virtual work, due to the rigid links among the points defining the local frame.

The internal forces can be calculated from the local part of the displacement, using EQ. 4.3.2.1 and taking into account the rigid links, for example, the fact that displacement u_E creates no strain;

$$\mathbf{f}_{int}(\mathbf{u}) = \mathbf{P}\mathbf{f}_{int\,L}(\mathbf{u}_L) = \mathbf{P}\,div_L[\boldsymbol{\sigma}_L(\mathbf{u}_L)] \quad \text{EQ.4.3.2.10}$$

Where, index L expresses that the coordinates and the spatial derivatives are taken in the local frame.

The virtual work of the internal forces is then:

$$\delta W_{int} = \int_\Omega \delta u^T P\, div_L[\sigma_L(u_L)] d\Omega \quad \text{EQ.4.3.2.11}$$

The integration by parts in the local frame introduces external surface forces f_{ext} :

$$\begin{aligned}\delta \mathbf{W}_{int} &= -\int_\Omega \boldsymbol{\sigma}_L(\mathbf{u}_L) : \boldsymbol{\varepsilon}_L\left(\mathbf{P}^T \delta \mathbf{u}\right) d\Omega + \int_\Gamma \delta \mathbf{u}^T \mathbf{P}\boldsymbol{\sigma}_L(\mathbf{u}_L).\mathbf{n}_L\, d\Omega \\ &= -\int_\Omega \boldsymbol{\sigma}_L(\mathbf{u}_L) : \boldsymbol{\varepsilon}_L\left(\mathbf{P}^T \delta \mathbf{u}\right) d\Omega + \int_\Gamma \delta \mathbf{u}^T \mathbf{f}_{ext}\, d\mathbf{S}\end{aligned} \quad \text{EQ.4.3.2.12}$$

Where Γ is the boundary of Ω, n_L is the normal to Γ expressed in the local frame.

To compute forces associated to the rigid links, first, new Lagrange multipliers are introduced to express the energy of a link:

$$W_{links} = \sum_{\substack{(I,J)\in(O,A,B,C)^2 \\ J>I}} \Lambda_{IJ} D_{IJ}(u_I, u_J) \qquad \text{EQ.4.3.2.13}$$

Where $D_{IJ} = \left\| \left(x_I^0 + u_I\right) - \left(x_J^0 + u_J\right) \right\| - \left\| x_I^0 - x_J^0 \right\|$ and x_I^0 are the initial coordinates of point *I* and the rigid link between points *I* and *J* is given by: $D_{IJ}(u_I, u_J) = 0$.

Then, the differentiation of this energy is used to obtain the virtual work to be introduced into the weak form of the equilibrium:

$$\delta W_{links} = -\sum_{\substack{(I,J)\in(O,A,B,C)^2 \\ J>I}} \delta\Lambda_{IJ} D_{IJ}(u_I, u_j) - \sum_{I\in(O,A,B,C)} \delta u_I \sum_{\substack{J\in(O,A,B,C) \\ J\neq I}} \delta\Lambda_{IJ} \frac{D_{IJ}(u_I, u_j)}{\partial u_I} \qquad \text{EQ.4.3.2.14}$$

$\forall \delta\mathbf{u}_I \quad I \in (O, A, B, C)$,

Note: The quantity $\sum_{\substack{J\in(O,A,B,C) \\ J\neq I}} \Lambda_{IJ} \frac{\partial D_{IJ}(u_I, u_J)}{\partial u_I} = F_{links\,IJ}$ can be viewed as the resisting force applied to point *I* to preserve the distances from this point to the other points of the local frame.

4.3.2.3 Weak form of equilibrium

Now, let us express the displacement field using EQ.4.3.2.1 and the local field projected on a Ritz basis:

$$\begin{aligned} \mathbf{u} &= \sum_{i=1}^{3} u_A^i \boldsymbol{\varphi}_X^i + u_B^i \boldsymbol{\varphi}_Y^i + u_C^i \boldsymbol{\varphi}_Z^i + u_O^i \boldsymbol{\varphi}_{1-X-Y-Z}^i + y^i \mathbf{P}\boldsymbol{\varphi}_L^i \\ &= \boldsymbol{\Phi}_P \hat{\mathbf{U}} \end{aligned} \qquad \text{EQ.4.3.2.15}$$

where $\varphi_X^i = Xe_i$, $\varphi_Y^i = Ye_i$, $\varphi_Z^i = Ze_i$, $\varphi_{1-X-Y-Z}^i = (1-X-Y-Z)e_i$, (e_1, e_2, e_3) is a basis of the global frame, $\{\varphi_L^i\}$ is a basis of local Ritz vectors obtained, for example, by finite element discretization or by modal analysis, $\hat{U}$ is the vector of the discrete unknowns:

$$\hat{U} = \begin{bmatrix} [u_A^i] \\ [u_B^i] \\ [u_C^i] \\ [u_O^i] \\ [y^i] \end{bmatrix} = \begin{bmatrix} \hat{U}_E \\ \hat{U}_L \end{bmatrix}, \text{ with } \hat{U}_E = \begin{bmatrix} [u_A^i] \\ [u_B^i] \\ [u_C^i] \\ [u_O^i] \end{bmatrix} \text{ and } \hat{U}_L = [y^i],$$

Φ_P is the projection basis: $\Phi_P = \left[\{\varphi_X^i\}, \{\varphi_Y^i\}, \{\varphi_Z^i\}, \{\varphi_{1-X-Y-Z}^i\}, \{P\varphi_L^i\}\right] = [\Phi_E, P\Phi_L]$.

EQ.4.3.2.5 and EQ.4.3.2.7 yield:

$$\begin{aligned} \dot{u} &= \Phi_P \dot{\hat{U}} \\ \ddot{u} &= \Phi_P \ddot{\hat{U}} + G\left(\dot{\hat{U}}\right) \\ \delta u &= \Phi_P \delta\hat{U} \end{aligned} \qquad \text{EQ.4.3.2.16}$$

where $G\left(\dot{\hat{U}}\right)$ is the gyroscopic contribution to the acceleration, given by:

$$G\left(\dot{\hat{U}}\right) = 2\dot{P}\left(\left[\dot{u}_A^i\right],\left[\dot{u}_B^i\right],\left[\dot{u}_C^i\right],\left[\dot{u}_O^i\right]\right)\Phi_L \dot{\hat{U}}_L$$

The final expression of the complete weak form of the dynamic equilibrium is obtained as:

$$\begin{aligned} \int_\Omega \delta\hat{U}^T\left(\Phi_P{}^T\right)\Phi_P \ddot{\hat{U}}\rho d\Omega + \int_\Omega \delta\hat{U}^T\left(\Phi_P{}^T\right)G\left(\dot{\hat{U}}\right)\rho d\Omega + \int_\Omega \sigma_L\left(\Phi_L \hat{U}_L\right) : \varepsilon_L\left(\Psi_P \delta\hat{U}\right) d\Omega \\ - \delta\Lambda^T D\left(\hat{U}_E\right) - \delta\hat{U}_E{}^T F_{links}\left(\Lambda, \hat{U}_E\right) = \int_\Gamma \delta\hat{U}^T\left(\Phi_P{}^T\right) f_{ext} dS \end{aligned} \qquad \text{EQ.4.3.2.17}$$

where $\Phi_L = \left\{\varphi_L^i\right\}$,

$\psi_P = \left[\left\{P^T\varphi_X^i\right\},\left\{P^T\varphi_Y^i\right\},\left\{P^T\varphi_Z^i\right\},\left\{P^T\varphi_{1-X-Y-Z}^i\right\},\left\{\varphi_L^i\right\}\right] = \left[P^T\Phi_E, \Phi_L\right]$

$\mathbf{D}$ is the vector formed by the 6 relations preserving the relative distances of points (O, A, B, C),

Λ is the vector of the Lagrange multipliers corresponding to each rigid link,

$\mathbf{F}_{\text{links}}$ is the vector of the link forces given by Equation (31).

EQ. 4.3.2.17 can be rewritten using classical matrix and vector operators obtained by finite element discretization:

$$\delta\hat{U}^T M_P \ddot{\hat{U}} + \delta\hat{U}^T F_{gyr}\left(\dot{\hat{U}}\right) + \delta\hat{U}^T K_L \hat{U}_L - \delta\Lambda^T D\left(\hat{U}_E\right) - \delta\hat{U}_E{}^T F_{links}\left(\Lambda, \hat{U}_E\right) = \delta\hat{U}^T F_{ext\,P} \qquad \text{EQ.4.3.2.18}$$

where $M_P = \overline{\Phi}_P{}^T M \overline{\Phi}_P$, with M being the classical mass matrix of sub-domain and $\overline{\Phi}_P$ being the projection matrix consisting of vectors of Φ_P discretized on the nodes of the mesh:

$$F_{gyr}\left(\dot{\hat{U}}\right) = \int_\Omega \left(\Phi_P^T\right) G\left(\dot{\hat{U}}\right) \rho d\Omega \qquad \text{EQ.4.3.2.19}$$

$K_L = \overline{\Psi}_P{}^T K \overline{\Phi}_L$, with K being the sub-domain's local stiffness matrix and $\overline{\Psi}_P$ and $\overline{\Phi}_L$ deduced (as was $\overline{\Phi}_L$) from $\overline{\Psi}_P$, $\overline{\Phi}_L$ and the mesh, $F_{ext\,P} = \overline{\Phi}_P{}^T F_{ext}$, with F_{ext} being the classical vector of the external forces assembled on the sub-domain.

Now, you are able to reduce the number of unknowns on the sub-domain drastically by choosing as the Ritz vectors, instead of classical finite element shape functions, an appropriate (and small) family of local reduction vectors. The modal vibration problem is purely local and guidelines found in the literature for the proper choice of the projection basis apply here.

Note 1

As far as inertia coupling with local vibration and overall large motion is concerned, two separate contributions must be considered. The first contribution appears in the projected mass matrix, which as now the following form:

$$\mathrm{M_P} = \begin{bmatrix} \mathrm{M_E} & \mathrm{M_C^T} \\ \mathrm{M_C} & \mathrm{M_V} \end{bmatrix} \qquad \text{EQ.4.3.2.20}$$

where M_E is the constant mass matrix corresponding only to the global displacement field given by $X\dot{u}_A + Y\dot{u}_B + Z\dot{u}_C + (1-X-Y-Z)\dot{u}_O$, M_V is the constant mass matrix corresponding to the local vibration given by u_L, M_C is a coupling matrix, variable with overall rotation, arising from the interaction between the local vibratory acceleration field expressed in the global frame $P\ddot{u}_L$ and the overall virtual displacement field $X\delta u_A + Y\delta u_B + Z\delta u_C + (1-X-Y-Z)\delta u_O$; $\mathrm{M_C^T}$ naturally comes from the symmetric interaction between virtual local displacement field and the overall acceleration field.

The second contribution to the inertia coupling is the gyroscopic forces.

Note 2

In RADIOSS a special procedure is used to linearize the rigid links. The method is fully explained in [109].

Note 3

The rigid body motion component of the displacement increment is computed in unconditionally stable way by the use of Lagrange Multiplier to impose the rigid links. The deforming part is generated by the local vibration modes retained in the reduction basis. Therefore, you can conclude that the stability condition is the same as that given by the local vibrating system. The critical time step is constant throughout the calculation and can be derived from the highest eigen frequency of the local reduced stiffness matrix with respect to the local reduced mass matrix.

The highest eigen frequency f_{max} is given by the system:

$$\Phi_L^T K \Phi_L V = \omega^2 \Phi_L^T M \Phi_L V \qquad \text{EQ.4.3.2.21}$$

where $\Phi_L = \{\varphi_L^i\}$ and $f = \dfrac{\omega}{2\pi}$.

Having determined f_{max}, the maximum time step which can be used on the reduced sub-domain while ensuring the stability of the time integration is:

$$\Delta t_{max} = \frac{1}{\pi f_{max}} \qquad \text{EQ.4.3.2.22}$$