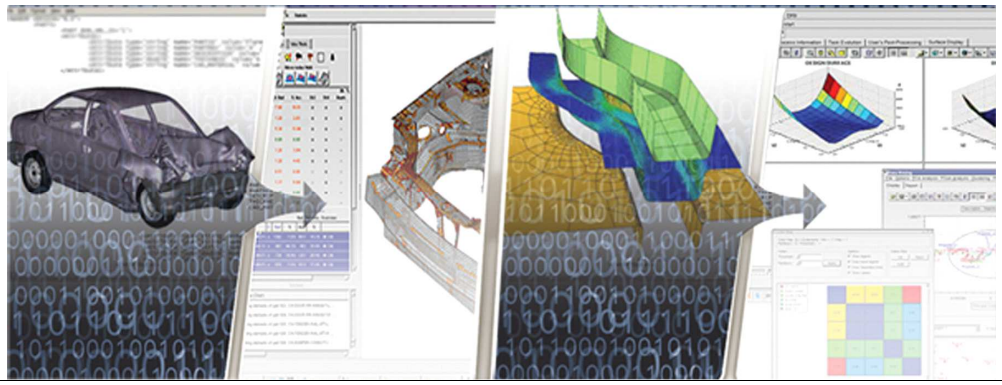

RADIOSS THEORY MANUAL

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Large Displacement Finite Element Analysis
Chapter 3

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Chapter 3

FINITE ELEMENT FORMULATION

3.0 FINITE ELEMENT FORMULATION

3.1 Finite Element Approximation

In the finite element method, the motion *x(X,t)* is approximated by:

$$x_i(X,t) = \Phi_I(X) x_{iI}(t) \qquad \text{EQ. 3.1.0.1}$$

where $\Phi_I(X)$ are the interpolating shape functions and x_{iI} is the position vector of node *I*. Summation over repeated indices is implied. In the case of lower indices, summation is over the number of space dimensions. For upper case indices, summation is over the number of nodes. The nodes in the sum depend on the type of entity considered. When the volume is considered, the summation is over all the nodes in the domain. When an element is considered, the sum is over the nodes of the element.

Similarly, nodal displacements are defined using EQ. 2.1.0.2 at nodes:

$$u_{iI}(X,t) = x_{iI}(t) - X_{iI} \qquad \text{EQ. 3.1.0.2}$$

The displacement field is:

$$u_i(X,t) = \Phi_I(X) u_{iI}(t) \qquad \text{EQ. 3.1.0.3}$$

The velocities are obtained by taking the material time derivative of the displacement giving:

$$v_i(X,t) = \frac{\partial u_i(X,t)}{\partial t} = \Phi_I(X) v_{iI}(t) \qquad \text{EQ. 3.1.0.4}$$

It is worth pointing out that the velocity is a material time derivative of displacements, i.e. the partial derivative with respect to time with the material coordinate fixed.

Finally, accelerations are similarly given by the material time derivative of velocities:

$$\dot{v}_i(X,t) = \Phi_I(X) \dot{v}_{iI}(t) \qquad \text{EQ. 3.1.0.5}$$

Emphasis is placed on the fact that shapes functions are functions of the material coordinates whatever the updated or the total Lagrangian formulation is used. All the dependency in the finite element approximation of the motion is taken into account in the values of the nodal variables.

From EQ. 2.4.1.5, the velocity gradient is given by:

$$L_{ij} = \frac{\partial v_i}{\partial x_j} = v_{iI} \frac{\partial \Phi_I}{\partial x_j} = v_{iI} \Phi_{I,j} \qquad \text{EQ. 3.1.0.6}$$

and the rate of deformation (EQ. 2.4.1.1) by:

$$D_{ij} = \frac{1}{2}(L_{ij} + L_{ji}) = \frac{1}{2}(v_{iI}\Phi_{I,j} + v_{jI}\Phi_{I,i}) \qquad \text{EQ. 3.1.0.7}$$

Similarly, the test functions are approximated as:

$$\delta v_i(X) = \Phi_I(X) \delta v_{iI} \qquad \text{EQ. 3.1.0.8}$$

where δv_{iI} are the virtual nodal velocities.

The test functions are next substituted into the principle of virtual power (EQ. 2.10.0.5) giving:

$$\delta v_{iI} \int_\Omega \frac{\partial \Phi_I}{\partial x_j} \sigma_{ji} d\Omega - \delta v_{iI} \int_\Omega \Phi_I \rho b_i d\Omega - \delta v_{iI} \int_{\Gamma_\sigma} \Phi_I \tau_i d\Gamma + \delta v_{iI} \int_\Omega \Phi_I \rho \dot{v}_i d\Omega = 0 \qquad \text{EQ. 3.1.0.9}$$

The virtual velocities must be kinematically admissible, i.e. satisfy boundary conditions on Γ_u, the part of the boundary where kinematical conditions are specified. Using the arbitrariness of the virtual nodal velocities everywhere except on Γ_u, the weak form of the momentum equation is:

$$\int_\Omega \frac{\partial \Phi_I}{\partial x_j}\sigma_{ji}d\Omega - \int_\Omega \Phi_I \rho b_i d\Omega - \int_{\Gamma_\sigma} \Phi_I \tau_i d\Gamma + \int_\Omega \Phi_I \rho \dot{v}_i d\Omega = 0 \qquad \text{EQ. 3.1.0.10}$$

with Γ_σ the part of the boundary where traction loads are imposed.

3.2 Internal and External Nodal Forces

As in Section 2.11, we define the nodal forces corresponding to each term in the virtual power equation.

The internal nodal forces are defined by:

$$\delta P^{\text{int}} = \delta v_{iI} f_{iI}^{\text{int}} = \int_\Omega \frac{\partial \delta v_i}{\partial x_j}\sigma_{ji}d\Omega = \delta v_{iI}\int_\Omega \frac{\partial \Phi_I}{\partial x_j}\sigma_{ji}d\Omega \qquad \text{EQ. 3.2.0.1}$$

The stress is the true (Cauchy) stress.

$$f_{iI}^{\text{int}} = \int_\Omega \sigma_{ji}\left(\frac{\partial \Phi_I}{\partial x_j}\right)d\Omega \qquad \text{EQ. 3.2.0.2}$$

These nodal forces are called internal because they represent the stresses in the body. The expression applies to both the complete mesh or to any element. It is pointed out that derivatives are taken with respect to spatial coordinates and that integration is taken over the current deformed configuration.

The external forces are similarly defined in terms of the virtual external power:

$$\delta P^{ext} = \delta v_{iI} f_{iI}^{ext} = \delta v_{il}\int_\Omega \Phi_I \rho b_i d\Omega + \delta v_{il}\int_{\Gamma_\sigma} \Phi_I \tau_i d\Gamma \qquad \text{EQ. 3.2.0.3}$$

so that external forces are given by:

$$f_{iI}^{ext} = \int_\Omega \Phi_I \rho b_i d\Omega + \int_{\Gamma_\sigma} \Phi_I \tau_i d\Gamma \qquad \text{EQ. 3.2.0.4}$$

3.3 Mass Matrix and Inertial Forces

The inertial body forces are defined by:

$$\delta P^{inert} = \delta v_{iI} f_{iI}^{inert} = \delta v_{iI}\int_\Omega \Phi_I \rho \dot{v}_i d\Omega = \delta v_{iI}\int_\Omega \Phi_I \rho \dot{v}_i d\Gamma \qquad \text{EQ. 3.3.0.1}$$

so that the inertia forces are given by:

$$f_{iI}^{inert} = \int_\Omega \Phi_I \rho \dot{v}_i d\Omega \qquad \text{EQ. 3.3.0.2}$$

or using the EQ. 3.1.0.5 for the accelerations:

$$f_{iI}^{inert} = \int_\Omega \rho \Phi_I \Phi_J d\Omega \dot{v}_{iJ} \qquad \text{EQ. 3.3.0.3}$$

It is usual to define the inertial nodal forces as the product of a mass matrix and the nodal accelerations. Defining the mass matrix as:

$$M_{ijIJ} = \delta_{ij} \int_{\Omega} \rho \Phi_I \Phi_J d\Omega$$ EQ. 3.3.0.4

the inertial forces are given by:

$$f_{iI}^{inert} = M_{ijIJ} \dot{v}_{jJ}$$ EQ. 3.3.0.5

3.4 Discrete Equations

Using the definitions of the internal and external forces, as well as the definition of the inertial forces, it is possible to write the weak form of the virtual power principle as:

$$\delta v_{iI} \left(M_{ijIJ} \dot{v}_{jJ} + f_{iI}^{\text{int}} - f_{iI}^{ext} \right) = 0$$ EQ. 3.4.0.6

or taking into account the arbitrariness of the virtual velocities:

$$M_{ijIJ} \dot{v}_{jJ} + f_{iI}^{\text{int}} = f_{iI}^{ext}$$ EQ. 3.4.0.7

3.5 Equation of Motion for Translational Velocities

EQ. 3.4.0.7 is written in matrix notation as:

$$M \frac{dv}{dt} = f^{ext} - f^{\text{int}}$$ EQ. 3.5.0.1

This is Newton's equation, where:

$$M = \int_{\Omega} \rho \Phi^T \Phi d\Omega$$ EQ. 3.5.0.2

is the mass matrix.

RADIOSS uses a lumped mass approach, i.e. each node represents a discrete mass of zero size. This creates a diagonal mass matrix *M*, eliminating, as we will see in Chapter 4, the need to solve simultaneous equations for the solution of nodal accelerations.

$$f^{ext} = \int_{\Gamma} \Phi^T \tau d\Gamma + \int_{\Omega} \rho \Phi^T b d\Omega$$ EQ. 3.5.0.3

is the externally applied load vector, and:

$$f_{iI}^{\text{int}} = \int_{\Omega} \sigma_{ij} \frac{\partial \Phi_I}{\partial x_j} d\Omega$$ EQ. 3.5.0.4

is the internal force vector.

Adding to the internal and external forces the anti-hourglass force vector and the contact force vector which will be described in the following chapters, we obtain the overall equation of motion:

$$M \frac{dv}{dt} = f^{ext} - f^{\text{int}} + f^{hgr} + f^{cont}$$ EQ. 3.5.0.5

3.6 Equation of Motion for Angular Velocities

Shell, beam and rigid body theory introduces nodal rotational degrees of freedom. The equations of motion for rotational degrees of freedom are complicated if written in the global reference frame. They are much simpler if written for each node in the principal reference frame attached to the node. The resulting equations are the

standard Euler equations. They are completely analogous to Newton's law governing translational degrees of freedom and are stated as follows:

$$l_1\alpha_1 + (I_3 - I_2)\omega_2\omega_3 = m_1^{ext} - m_1^{\text{int}} \quad \text{EQ. 3.6.0.1}$$

$$l_2\alpha_2 + (I_1 - I_3)\omega_1\omega_3 = m_2^{ext} - m_2^{\text{int}} \quad \text{EQ. 3.6.0.2}$$

$$l_3\alpha_3 + (I_2 - I_1)\omega_1\omega_2 = m_3^{ext} - m_3^{\text{int}} \quad \text{EQ. 3.6.0.3}$$

where

- I_1, I_2, I_3 are the principal moments of inertia about the x, y and z axes respectively,
- $\boldsymbol{\alpha 1, \alpha 2, \alpha 3}$ are the angular accelerations expressed in the principal reference frame,
- $\boldsymbol{\omega 1, \omega 2, \omega 3}$ are the angular velocities,
- $m_1^{ext}, m_2^{ext}, m_3^{ext}$ are the principal externally applied moments,
- $m_1^{\text{int}}, m_2^{\text{int}}, m_3^{\text{int}}$ are the principal internal moments.

The equation of motion for rotational degrees of freedom is thus very similar to that for translational degrees of freedom. In matrix notation and in the nodal principal reference frame:

$$I\frac{d\omega}{dt} = M^{ext} - M^{\text{int}} + F(\omega) \quad \text{EQ. 3.6.0.4}$$

The vector function $F(\omega)$ is computed for a value of ω at $t - \delta t/2$. Equation 3.6.0.4 is used for rigid body motion.

For shell, beam and spring using a spherical inertia, the equation of motion becomes:

$$I\frac{d\omega}{dt} = M^{ext} - M^{\text{int}} + M^{hgr} \quad \text{EQ. 3.6.0.5}$$

where:

$I = \sum_{elements} I_e$ is the diagonal inertia matrix,

$M^{ext} = \sum_{elements} m^{ext}$ is the externally applied moment vector,

$M^{\text{int}} = \sum_{elements} m^{\text{int}}$ is the internal moment vector,

$M^{hgr} = \sum_{shells} m^{hgr}$ is the anti-hourglass shell moment vector.

3.7 Element Coordinates

Finite elements are usually developed with shape functions expressed in terms of an intrinsic coordinates system ξ, η, ζ. It is shown hereafter that expressing the shape functions in terms of intrinsic coordinates is equivalent to using material coordinates.

When an element is treated in terms of intrinsic coordinates, we are concerned with three domains that correspond to this element:

- the domain in the intrinsic coordinates system,
- the current element domain,

- the initial reference element domain.

ξ is associated with the direction u

η is associated with the direction v

ζ is associated with the direction w

The motion in each element can thus be described by the composition of three maps (the reasoning is described only for the direction u):

- the map from the intrinsic coordinates system to the initial configuration:

$$X(\xi) \qquad \text{EQ. 3.7.0.1}$$

- the map from the intrinsic coordinates system to the current configuration:

$$x(\xi,t) \qquad \text{EQ. 3.7.0.2}$$

- the map from the initial to the current configuration:

$$x=\varphi(X,t) \qquad \text{EQ. 3.7.0.3}$$

So, it is possible to approximate the motion in an element by:

$$x_i(\xi,t)=\Phi_I(\xi)x_{iI}(t) \qquad \text{EQ. 3.7.0.4}$$

Shape functions $\Phi_I(\xi)$ have no dimensions. They simply relate coordinates in the physical world to the intrinsic coordinates system. Writing EQ. 3.7.0.4 at t=0 , we obtain:

$$x_i(\xi)=x_i(\xi,0)=\Phi_I(\xi)x_{iI}(0)=\Phi_I(\xi)x_{iI} \qquad \text{EQ. 3.7.0.5}$$

So, it can be seen from the last equation that the material coordinates system and the intrinsic coordinates system are invariant in a Lagrangian element. As a result, as intrinsic coordinates are time invariant and it is possible to write displacements, velocities and accelerations in terms of intrinsic coordinates (one coordinate system, the two other coordinates have similar shape functions):

$$u_i(\zeta,t)=\Phi_I(\xi)u_{iI}(t) \qquad \text{EQ. 3.7.0.6}$$

$$\dot{u}_i(\zeta,t)=\Phi_I(\xi)\dot{u}_{iI}(t) \qquad \text{EQ. 3.7.0.7}$$

$$\dot{v}_i(\zeta,t)=\Phi_I(\xi)\dot{v}_{iI}(t) \qquad \text{EQ. 3.7.0.8}$$

Isoparametric elements use the same shape functions for the interpolation of $x,u,\dot{u}$ and $\dot{v}$.

3.8 Integration and Nodal Forces

In practice, integrals over the current domain in the definition of the internal nodal forces (EQ. 3.5.0.4), of the external nodal forces (EQ. 3.5.0.3) and of the mass matrix have to be transformed into integrals over the domain in the intrinsic coordinate system Δ.

Using EQ. 2.3.0.5, integrals on the current configuration are related to integrals over the reference configuration and over the domain in the intrinsic coordinate system by:

$$\int_\Omega g(x)d\Omega=\int_{\Omega^0} g(x)|F|d\Omega_0=\int_\Delta g(\xi)|F_\xi|d\Delta \qquad \text{EQ. 3.8.0.1}$$

and

$$\int_{\Omega_0} g(X)d\Omega_0 = \int_{\Delta} g(\xi)\left|F_{\xi}^{\ 0}\right|d\Delta$$ EQ. 3.8.0.2

where:

$$\left|F\right|$$ EQ. 3.8.0.3

is the Jacobian determinant of the transformation between the current and the initial configuration,

$$\left|F_{\xi}\right|$$ EQ. 3.8.0.4

is the Jacobian determinant of the transformation between the current configuration and the domain in the intrinsic coordinate system and,

$$\left|F_{\xi}^{\ 0}\right|$$ EQ. 3.8.0.5

is the Jacobian determinant of the transformation between the reference configuration and the intrinsic coordinate system.

On the other hand, it comes from EQ. 2.3.0.2 and EQ. 3.7.0.4:

$$F_{\xi kj} = \frac{\partial x_k}{\partial \xi_j} = \frac{\partial \Phi_I(\xi)}{\partial \xi_j} x_{kI}$$ EQ. 3.8.0.6

So, using EQ. 3.8.0.1, internal forces computed by integration over the current domain will be obtained by the following quadrature:

$$f_{iI}^{\text{int}} = \int_{\Omega} \sigma_{ij} \frac{\partial \Phi_I}{\partial x_j} d\Omega = \int_{\Delta} \sigma_{ij} \frac{\partial \Phi_I}{\partial x_j} \left|F_{\xi}\right| d\Delta$$ EQ. 3.8.0.7

and $\left|F_{\xi}\right|$ obtained from EQ. 3.8.0.6.

External forces and the mass matrix can similarly be integrated over the domain in the intrinsic coordinate system.

3.9 Derivatives of Functions

The definition of internal forces also shows that derivatives of the form:

$$\frac{\partial}{\partial x_j}$$ EQ. 3.9.0.1

need to be computed. These spatial derivatives are obtained by implicit differentiation. Considering the velocity gradient e.g.:

$$L_{ij} = \frac{\partial v_i}{\partial x_j}$$ EQ. 3.9.0.2

one has:

$$L_{ij} = \frac{\partial v_i}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_j} = \frac{\partial v_i}{\partial \xi_k} F_{\xi kj}^{-1} = v_{iI} \frac{\partial \Phi_I}{\partial \xi_k} F_{\xi kj}^{-1}$$ EQ. 3.9.0.3

where:

$$F_{\xi kj} = \frac{\partial x_k}{\partial \xi_j} = \frac{\partial \Phi_I(\xi)}{\partial \xi_j} x_{kI}$$ EQ. 3.9.0.4

is the Jacobian matrix of the map between the current coordinates and the intrinsic coordinates.

Usually, it is not possible to have closed form expression of the Jacobian matrix. As a result the inversion will be performed numerically and numerical quadrature will be necessary for the evaluation of nodal forces.

3.10 Numerical Quadrature - Reduced Integration

All elements in RADIOSS are integrated numerically. Hence, the integrals for nodal forces are replaced by a summation:

$$\int f(\xi)d\xi = \sum_{j=1}^{n} w_j f(\xi_j) \qquad \text{EQ. 3.10.0.1}$$

where *n* is the number of integration points in the element and w_j is the weight associated to the integration point *j*. Values of w_j and locations of ξ_j are given in tables according to the numerical quadrature approach. RADIOSS uses either full or reduced integration schemes.

For full integration, the number of integration points is sufficient for the exact integration of the virtual work expression. The full integration scheme is often used in programs for static or dynamic problems with implicit time integration. It presents no problem for stability, but sometimes involves "locking" and the computation is often expensive.

Reduced integration can also be used. In this case, the number of integration points is sufficient for the exact integration of the contributions of the strain field that are one order less than the order of the shape functions. The incomplete higher order contributions to the strain field present in these elements are not integrated.

The reduced integration scheme, especially with one-point quadrature is widely used in programs with explicit time integration to compute the force vectors. It drastically decreases the computation time, and is very competitive if the spurious singular modes (often called “hourglass modes” which result from the reduced integration scheme) are properly stabilized. In two dimensions, a one point integration scheme will be almost four times less expensive than a four point integration scheme. The savings are even greater in three dimensions. The use of one integration point is recommended to save CPU time, but also to avoid "locking" problems, e.g. shear locking or volume locking.

Shear locking is related to bending behavior. In the stress analysis of relatively thin members subjected to bending, the strain variation through the thickness must be at least linear, so constant strain first order elements are not well suited to represent this variation, leading to shear locking. Fully integrated first-order isoparametric elements (tetrahedron) also suffer from shear locking in the geometries where they cannot provide the pure bending solution because they must shear at the numerical integration points to represent the bending kinematic behavior. This shearing then locks the element, i.e. the response is far too stiff.

On the other hand, most fully integrated solid elements are unsuitable for the analysis of approximately incompressible material behavior (volume locking). The reason for this is that the material behavior forces the material to deform approximately without volume changes. Fully integrated solid elements, and in particular low-order elements do not allow such deformations. This is another reason for using selectively reduced integration. Reduced integration is used for volume strain and full integration is used for the deviatoric strains.

However, as mentioned above, the disadvantage of reduced integration is that the element can admit deformation modes that are not causing stresses at the integration points. These zero-energy modes make the element rank-deficient which causes a phenomenon called hour-glassing; the zero-energy modes start propagating through the mesh, leading to inaccurate solutions. This problem is particularly severe in first-order quadrilaterals and hexahedra.

To prevent these excessive deformations, a small artificial stiffness or viscosity associated with the zero-energy deformation modes is added, leading in EQ. 3.5.0.1 and. EQ. 3.6.0.4 to anti-hourglass force and moment vectors:

$$M\frac{dv}{dt} = f^{ext} - f^{\text{int}} + f^{hgr} \qquad \text{EQ. 3.10.0.2}$$

$$I\frac{d\omega}{dt} = M^{ext} - M^{\text{int}} + M^{hgr} \qquad \text{EQ. 3.10.0.3}$$

Zero-energy or hourglass modes are controlled using a perturbation stabilization as described by Flanagan-Belytschko [12], or physical stabilization as described in [15] (Chapter 5).

So, for isoparametric elements, reduced integration allows simple and cost effective computation of the volume integrals, in particular on vectorized supercomputers, and furnishes a simple way to cope with locking aspects, but at the cost of allowing hour-glassing.

3.11 Numerical Procedures

The RADIOSS numerical solver can be summarized by the flow chart in Figure 3.11.1. For each time step in a particular analysis, the algorithm used to compute results is:

1. For the displacement, velocity and acceleration at a particular time step, the external force vector is constructed and applied.

2. A loop over element is performed, in which the internal and hourglass forces are computed, along with the size of the next time step. The procedure for this loop is:

   2a. The Jacobian matrix is used to relate displacements in the intrinsic coordinates system to the physical space:

   $$\left.\frac{\partial \Phi}{\partial x_j}\right|_t = F_\xi^{-1} \left.\frac{\partial \Phi}{\partial \xi}\right|_t \qquad \text{EQ. 3.11.0.4}$$

   2b. The strain rate is calculated:

   $$\dot{\varepsilon}_{ij} = \left(\frac{\partial \Phi_I}{\partial x_j}\right)\dot{x} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad \text{EQ. 3.11.0.5}$$

   2c. The stress rate is calculated:

   $$\dot{\sigma}_{ij} = f\left(\dot{\varepsilon}, material - law\right) \qquad \text{EQ. 3.11.0.6}$$

   2d. Cauchy stresses are computed using explicit time integration:

   $$\sigma\left(t + \Delta t\right) = \sigma\left(t\right) + \dot{\sigma}\Delta t \qquad \text{EQ. 3.11.0.7}$$

   2e. The internal and hourglass force vectors are computed.

   2f. The next time step size is computed, using element or nodal time step methods (Chapter 4.)

3. After the internal and hourglass forces are calculated for each element, the algorithm proceeds by computing the contact forces between any interfaces.

4. With all forces known, the new accelerations are calculated using the mass matrix and the external and internal force vectors:

   $$\dot{v}_i = M^{-1}\left(f_{ext_i} - f_{\mathrm{int}_i}\right) \qquad \text{EQ. 3.11.0.8}$$

5. Finally, time integration of velocity and displacement is performed using the new value.

Figure 3.11.1 Numerical Procedure [1]