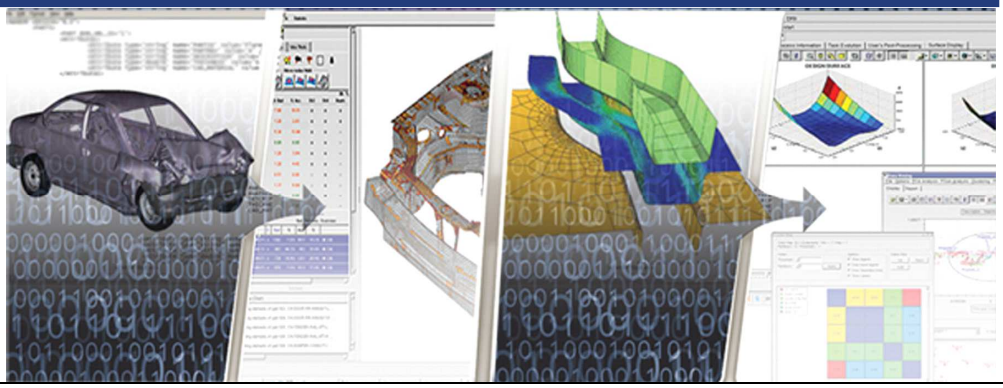

RADIOSS THEORY MANUAL
Version 2017 – January 2017
SPH

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Chapter 3

SMOOTH PARTICLE HYDRODYNAMICS

3.0 SMOOTH PARTICLE HYDRODYNAMICS

Smooth Particle Hydrodynamics (SPH) is a meshless numerical method based on interpolation theory. It allows any function to be expressed in terms of its values at a set of disordered point's so-called particles. SPH is not based on the particle physics theory. The conservation laws of continuum dynamics, in the form of partial differential equations, are transformed into integral equations through the use of kernel approximation. A comprehensive state-of-the-art of the method is given in [78], [81], and [82]. These techniques were initially developed in astrophysics [79] and [80]. During the 1991-1995 periods, SPH has become widely recognized and has been used extensively for fluid and solid mechanics type of applications. SPH method is implemented in RADIOSS in Lagrangian approach whereby the motion of a discrete number of particles is followed in time.

SPH is a complementary approach with respect to ALE method. When the ALE mesh is too distorted to handle good results (for example in the case of vortex creation), SPH method allows getting a sufficiently accurate solution.

3.1 SPH approximation of a function

Let $\prod f(x)$ the integral approximation of a scalar function *f* in space:

$$\prod f(x) = \int_{\Omega} f(y) W(x-y,h) dy \qquad \text{EQ.3.1.0.1}$$

with *h* the so-called smoothing length and *W* a kernel approximation such that:

$$\forall x, \int_{\Omega} W(x-y,h) dy = 1 \qquad \text{EQ. 3.1.0.2}$$

and $\forall x, \lim_{h\to 0} W(x-y,h) = \delta(x-y)$ (in a suitable sense) EQ. 3.1.0.3

δ denotes the Dirac function.

Let a set of particles i=1, n at positions x_i (i=1,n) with mass m_i and density ρ_i. The smoothed approximation of the function *f* is (summation over neighbouring particles and the particle *i* itself):

$$\prod_s f(x) = \sum_{i=1,n} \frac{m_i}{\rho_i} f(x_i) W(x-y,h) \qquad \text{EQ. 3.1.0.4}$$

The derivatives of the smoothed approximation are obtained by ordinary differentiation.

$$\nabla f(x) = \sum_{i=1,n} \frac{m_i}{\rho_i} f(x_i) \nabla W(x-y,h) \qquad \text{EQ. 3.1.0.5}$$

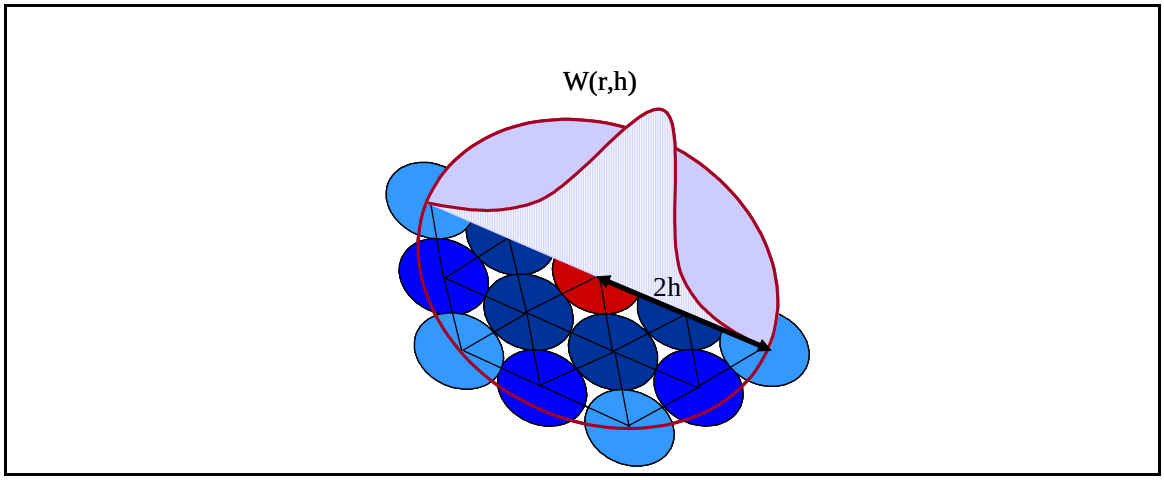
The following kernel [83] which is an approximation of Gaussian kernel by cubic splines was chosen (Figure 3.1.1):

$$r \le h \Rightarrow W(r,h) = \frac{3}{2\pi h^3}\left[\frac{2}{3} - \left(\frac{r}{h}\right)^2 + \frac{1}{2}\left(\frac{r}{h}\right)^3\right] \qquad \text{EQ. 3.1.0.6}$$

$$h \le r \le 2h \Rightarrow W(r,h) = \frac{1}{4\pi h^3}\left(2 - \frac{r}{h}\right)^3 \qquad \text{EQ. 3.1.0.7}$$

and $2h \le r \Rightarrow W(r,h) = 0$ EQ. 3.1.0.8

Figure 3.1.1 Kernel based on spline functions



This kernel has compact support, so that for each particle *i*, only the closest particles contribute to approximations at *i* (this feature is computationally efficient). The accuracy of approximating EQ. 3.1.0.1 by EQ. 3.1.0.4 depends on the order of the particles.

3.2 Corrected SPH approximation of a function

Corrected SPH formulation [85], [86] has been introduced in order to satisfy the so-called consistency conditions:

$$\int_{\Omega} W(y-x,h)=1, \forall x \qquad \text{EQ. 3.2.0.1}$$

$$\int_{\Omega} (y-x)W(y-x,h)=0, \forall x \qquad \text{EQ. 3.2.0.2}$$

These equations insure that the integral approximation of a function *f* coincides with *f* for constant and linear functions of space.

CSPH is a correction of the kernel functions:

$$\hat{W}_j(x,h)=W_j(x,h)\alpha(x)\left[1+\beta(x)\bullet\left(x-x_j\right)\right], \text{ with } W_j(x,h)=W\left(x-x_j,h\right) \qquad \text{EQ. 3.2.0.3}$$

where the parameters $\alpha(x)$ and $\beta(x)$ are evaluated by enforcing the consistency condition, now given by the point wise integration as:

$$\sum_j V_j\hat{W}_j(x,h)=1, \forall x \qquad \text{EQ. 3.2.0.4}$$

$$\sum_j V_j\left(x-x_j\right)\hat{W}_j(x,h)=0, \forall x \qquad \text{EQ. 3.2.0.5}$$

These equations enable the explicit evaluation of the correction parameters $\alpha(x)$ and $\beta(x)$ as follows:

$$\beta(x)=\left[\sum_j V_j(x-x_j)\otimes(x-x_j)W_j(x,h)\right]^{-1}\sum_j V_j(x_j-x)W_j(x,h) \quad \text{EQ. 3.2.0.6}$$

$$\alpha(x)=\frac{1}{\sum_j V_j W_j(x,h)\left[1+\beta(x)\bullet(x-x_j)\right]} \quad \text{EQ. 3.2.0.7}$$

Since the evaluation of gradients of corrected kernel (which are used for the SPH integration of continuum equations) becomes very expensive, corrected SPH limited to order 0 consistency has been introduced. Therefore, the kernel correction reduces to the following equations:

$$\hat{W}_j(x,h)=W_j(x,h)\alpha(x) \quad \text{EQ. 3.2.0.8}$$

$$\sum_j V_j\hat{W}_j(x,h)=1,\forall x \quad \text{EQ. 3.2.0.9}$$

that is $\alpha(x)=\dfrac{1}{\sum_j V_j W_j(x,h)}$ EQ. 3.2.0.10

Note that SPH corrections generally insure a better representation even if the particles are not organized into a hexagonal compact net, especially close to the integration domain frontiers. SPH corrections also allow the smoothing length *h* to values different to the net size Δx to be set.

3.3 SPH Integration of continuum equations

In order to keep an almost constant number of neighbors contributing at each particle, use smoothing length varying in time and in space.

Consider d_i the smoothing length related to particle *i*;

$W_j(i)=\hat{W}\left(x_i-x_j,\dfrac{d_i+d_j}{2}\right)$ and $\nabla W_j(i)=grad\left|_{x_i}\left[\hat{W}\left(x-x_j,\dfrac{d_i+d_j}{2}\right)\right]\right.$ if kernel correction,

EQ. 3.3.0.1

or $W_j(i)=W\left(x_i-x_j,\dfrac{d_i+d_j}{2}\right)$ and $\nabla W_j(i)=grad\left|_{x_i}\left[W\left(x-x_j,\dfrac{d_i+d_j}{2}\right)\right]\right.$ without kernel correction.

EQ. 3.3.0.2

At each time step, density is updated for each particle *i*, according to:

$$\left.\frac{d\rho}{dt}\right|_i=-\rho_i\nabla\cdot v\big|_i \quad \text{EQ. 3.3.0.3}$$

with $\nabla\cdot v\big|_i=\sum\dfrac{m_j}{\rho_j}(v_i-v_j)\cdot\nabla W_j(i)$ EQ. 3.3.0.4

where, m_j indicates the mass of a particle *i*, ρ_i its density, v_i its velocity.

Strain tensor is obtained by the same way when non pure hydrodynamic laws are used or in the other words when law uses deviatoric terms of the strain tensor:

$$\left.\frac{dv^\alpha}{dx^\beta}\right|_i = \sum \frac{m_j}{\rho_j}\left(v_i^\alpha - v_j^\alpha\right)\frac{dW_j}{dx^\beta}(i), \alpha = 1...3, \beta = 1...3. \qquad \text{EQ. 3.3.0.5}$$

Next the constitutive law is integrated for each particle. Then Forces are computed according to :

$$m_i \left.\frac{dv}{dt}\right|_i = -\sum_j V_i V_j \left[p_i \nabla W_j(i) - p_j \nabla W_i(j)\right] - \sum_j m_i m_j \pi_{ij} \frac{\left[\nabla W_j(i) - \nabla W_i(j)\right]}{2} \qquad \text{EQ. 3.3.0.6}$$

where p_i and p_j are pressures at particles *i* and *j*, and π_{ij} is a term for artificial viscosity. The expression is more complex for non pure hydrodynamic laws. Note that the previous equation reduces to the following one when there is no kernel correction:

$$m_i \left.\frac{dv}{dt}\right|_i = -\sum_j V_i V_j \left[p_i + p_j\right] \nabla W_j(i) - \sum_j m_i m_j \pi_{ij} \nabla W_j(i), \text{ since } \nabla W_i(j) = -\nabla W_j(i) \qquad \text{EQ. 3.3.0.7}$$

Then, search distances are updated according to:

$$\frac{d(d_i)}{dt} = d_i \frac{\nabla \cdot v|_i}{3} \qquad \text{EQ. 3.3.0.8}$$

in order particles to keep almost a constant number of neighbors into their kernels (ρd^3 is kept constant).

3.4 Artificial viscosity

As usual in SPH implementations [83], viscosity is rather an inter-particles pressure than a bulk pressure. It was shown that the use of EQ. 3.4.0.1 and EQ. 3.4.0.2 generates a substantial amount of entropy in regions of strong shear even if there is no compression.

$$\pi_{ij} = \frac{-q_b \frac{c_i + c_j}{2} \mu_{ij} + q_\alpha \mu_{ij}^2}{\frac{(\rho_i + \rho_j)}{2}} \qquad \text{EQ. 3.4.0.1}$$

$$\text{with } \mu_{ij} = \frac{d_{ij}\left(v_i - v_j\right) \bullet \left(X_i - X_j\right)}{\left\|X_i - X_j\right\|^2 + \varepsilon d_{ij}^2} \qquad \text{EQ. 3.4.0.2}$$

where X_i (resp. X_j) indicates the position of particle I (resp.j) and c_i (resp c_j) is the sound speed at location *i* (resp.j), and q_a and q_b are constants. This leads us to introduce EQ. 3.4.0.3 and EQ. 3.4.0.4, as explained in [82]. The artificial viscosity is decreased in regions where vorticity is high with respect to velocity divergence.

$$\pi_{ij} = \frac{-q_b \frac{c_i + c_j}{2} \mu_{ij} + q_\alpha \mu_{ij}^2}{\frac{(\rho_i + \rho_j)}{2}} \qquad \text{EQ. 3.4.0.3}$$

with $\mu_{ij} = \dfrac{d_{ij}\left(v_i - v_j\right)\bullet\left(X_i - X_j\right)}{\left\|X_i - X_j\right\|^2 + \varepsilon d_{ij}^2}\dfrac{\left(f_i + f_j\right)}{2}$, $f_k = \dfrac{\left\|\nabla\cdot v\right|_k\|}{\left\|\nabla\cdot v\right|_k\| + \left\|\nabla\times v\right|_k\| + \varepsilon'\dfrac{c_k}{d_k}}$ EQ. 3.4.0.4

Default values for q_a and q_b are respectively set to 2 and 1.

3.5 Stability: Time step control

The stability conditions of explicit scheme in SPH formulation can be written over cells or on nodes.

3.5.1 Cell time step

In case of cell stability computation (when no nodal time step is used), the stable time step is computed as:

$$\Delta t = \Delta t_{sca}\cdot\min_i\left(\frac{d_i}{c_i\left(\alpha_i + \sqrt{\alpha_i^2 + 1}\right)}\right), with \quad \alpha_i = \left(q_b + \frac{q_a\cdot\bar{\mu}_i\cdot d_i}{c_i}\right), and \quad \bar{\mu}_i = \max_j\left(\mu_{ij}\right)$$

EQ. 3.5.1.1

Δt_{sca} is the user defined coefficient (RADIOSS option /DT or /DT/SPHCEL). The value of Δt_{sca} =0.3 is recommended in [83].

3.5.2 Nodal time step

In case of nodal time step, stability time step is computed in a more robust way:

$$\Delta t_i = \sqrt{\frac{2m_i}{K_i}} \text{ at particle i}$$ EQ. 3.5.2.1

Use the following notations, if kernel correction:

$$W_j(i) = \hat{W}\left(x_i - x_{j'}\frac{d_i + d_j}{2}\right) and\ \nabla W_j(i) = grad\left|_{xi}\left[\hat{W}\left(x - x_{j'}\frac{d_i + d_j}{2}\right)\right]\right.$$ EQ. 3.5.2.2

Or, if no kernel correction:

$$W_j(i) = W\left(x_i - x_{j'}\frac{d_i + d_j}{2}\right) and\ \nabla W_j(i) = grad\left|_{xi}\left[W\left(x - x_{j'}\frac{d_i + d_j}{2}\right)\right]\right.$$ EQ. 3.5.2.3

Recalling that apart from the artificial viscosity terms:

$$F_i = \sum_j F_{ij}, F_{ij} = V_iV_j\left[p_i\nabla W_j(i) - p_j\nabla W_j(j)\right]$$ EQ. 3.5.2.4

write $\left|K_{ij}\right| = \left\|\dfrac{dF_{ij}}{d\left(u_i - u_j\right)}\right\| \le \dfrac{d}{d\left(u_i - u_j\right)}\left(V_iV_j\left[p_i\left\|\nabla W_j(i)\right\| + p_j\left\|\nabla W_i(j)\right\|\right]\right)$ EQ. 3.5.2.5

where u_i - u_j is the relative displacement of particles i and j. Keeping the only first order terms leads to :

$$\left|K_{ij}\right| \le V_i V_j \left[\frac{dp_i}{d\left(u_i - u_j\right)} \left\|\nabla W_j(i)\right\| + \frac{dp_j}{d\left(u_i - u_j\right)} \left\|\nabla W_i(j)\right\| \right] \quad \text{EQ.3.5.2.6}$$

Where,

$$V_i V_j \frac{dp_i}{d\left(u_i - u_j\right)} \left\|\nabla W_j(i)\right\| = V_i V_j \frac{dp_i}{d\rho_i} \cdot \frac{d\rho_i}{d\left(u_i - u_j\right)} \left\|\nabla W_j(i)\right\| = V_i V_j c_i^2 \frac{d\rho_i}{d\left(u_i - u_j\right)} \left\|\nabla W_j(i)\right\| \quad \text{EQ.3.5.2.7}$$

that is $V_i V_j \frac{dp_i}{d\left(u_i - u_j\right)} \left\|\nabla W_j(i)\right\| = m_i c_i^2 \dot{V}_j^2 \left\|\nabla W_j(i)\right\|^2$ EQ.3.5.2.8

Same reasoning leads to:

$$V_i V_j \frac{dp_j}{d\left(u_i - u_j\right)} \left\|\nabla W_i(j)\right\| = m_j c_j^2 \dot{V}_i^2 \left\|\nabla W_i(j)\right\|^2 \quad \text{EQ.3.5.2.9}$$

So that $\left|K_{ij}\right| \le m_i c_i^2 \dot{V}_j^2 \left\|\nabla W_j(i)\right\|^2 + m_j c_j^2 \dot{V}_i^2 \left\|\nabla W_i(j)\right\|^2$ EQ.3.5.2.10

Stiffness around node *i* is then estimated as:

$$\left|K_i\right| \le \sum_j \left|K_{ij}\right| \quad \text{EQ.3.5.2.11}$$

3.6 Conservative smoothing of velocities

It can be shown that the SPH method is unstable in tension. The instability is shown to result from an effective stress with a negative modulus (imaginary sound speed) being produced by the interaction between the constitutive relation and the kernel function, and is not caused by the numerical time integration algorithm [84]. According to [82], use special filtering of velocities (so called conservative smoothing, because momentum quantities are not modified):

$$V_i(smoothed) = v_i + \alpha_{cs} \sum_j \frac{2m_j}{\rho_i + \rho_j} \left(v_j - v_i\right) \frac{W_i(j) + W_j(i)}{2} \quad \text{EQ.3.6.0.1}$$

3.7 SPH cell distribution

It is recommended to distribute the particles through a hexagonal compact or a cubic net.

3.7.1 Hexagonal compact net

A cubic centered faces net realizes a hexagonal compact distribution and this can be useful to build the net (Figure 3.7.1). The nominal value h_0 is the distance between any particle and its closest neighbor. The mass of the particle m_p may be related to the density of the material ρ and to the size h_0 of the hexagonal compact net, with respect to the following equation:

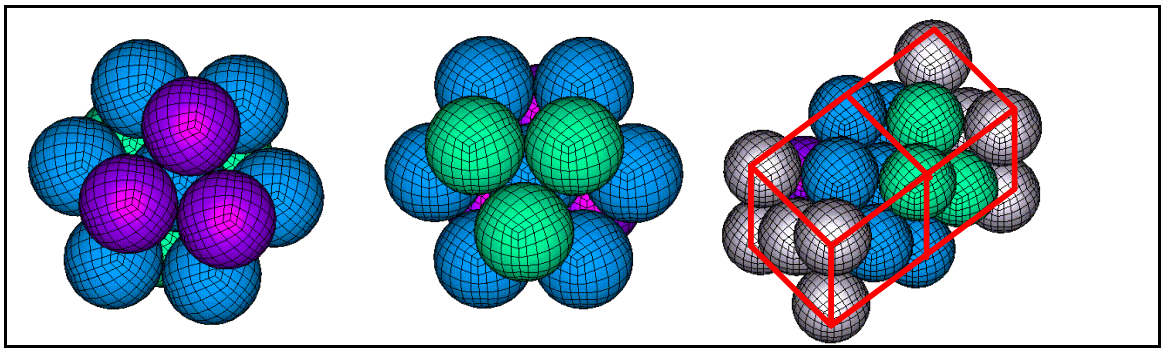
$$m_p \approx \frac{h_0^3}{\sqrt{2}} \rho \quad \text{EQ.3.7.1.1}$$

since the space can be partitioned into polyhedras surrounding each particle of the net, each one with a volume:

$$V_p \approx \frac{h_0^3}{\sqrt{2}} \qquad \text{EQ.3.7.1.2}$$

But, due to discretization error at the frontiers of the domain, mass consistency better corresponds to $m_P = \frac{\rho V}{n}$ where V is the total volume of the domain and n the number of particles distributed in the domain.

Figure 3.7.1 Local view of hexagonal compact net and perspective view of cubic centered faces net



Note that choosing h_0 for the smoothing length insures naturally consistency up to order 1 if the previous equation is satisfied.

Weight functions vanish at distance 2h where h is the smoothing length. In an hexagonal compact net with size h_0, each particle has exactly 54 neighbors within the distance $2h_0$ (Table 3.7.1).

Table 3.7.1 Number of neighbors in a hexagonal compact net

Distance d	Number of particles at distance d	Number of particles within distance d
h_0	12	12
$\sqrt{2}h_0$	6	18
$\sqrt{3}h_0$	24	42
$2h_0$	12	**54**
$\sqrt{5}h_0$	24	78

3.7.2 Cubic net

Let *c* the side length of each elementary cube into the net. The mass of the particles m_p should be related to the density of the material ρ and to the size *c* of the net, with respect to the following equation:

$$m_p \approx c^3 \rho \qquad \text{EQ.3.7.1.3}$$

By experience, a larger number of neighbors must be taken into account with the hexagonal compact net, in order to solve the tension instability as explained in following sections. A value of the smoothing length between *1.25c* and *1.5c* seems to be suitable. In the case of smoothing length *h=1.5c*, each particle has 98 neighbors within the distance 2h.

Table 3.7.2 Number of neighbors in a cubic net

Distance d	Number of particles at distance d	Number of particles within distance d
c	6	6
$\sqrt{2}c$	12	18
$\sqrt{3}c$	8	26
2c	6	32
$\sqrt{5}c$	24	56
$\sqrt{6}c$	24	80
$2\sqrt{2}c$	12	92
3c	6	**98**