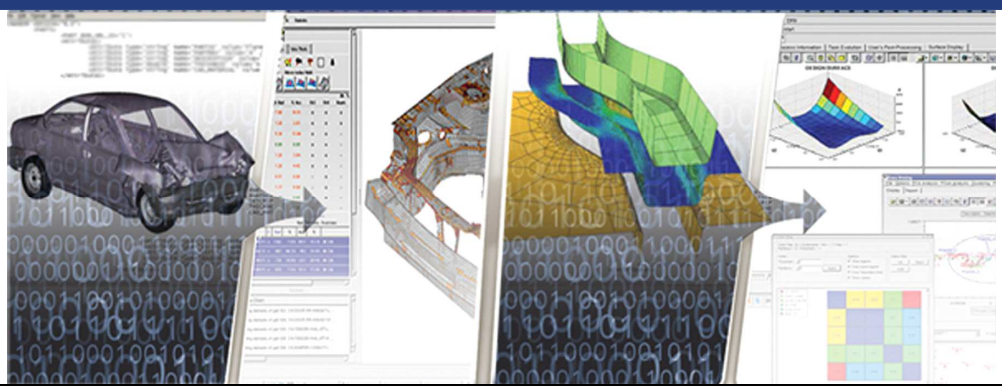

RADIOSS THEORY MANUAL

Version 2017 – January 2017

Large Displacement Finite Element Analysis

Chapter 11

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Chapter 11

STATIC

11.0 STATIC

Explicit scheme is generally used for time integration in RADIOSS, in which velocities and displacements are obtained by direct integration of nodal accelerations as described in Chapter 4. With this approach, the time step is often small due to stability condition. For the static solution of structural mechanical problems as the steady state is a part of the transient response for a temporal-step load, the use of explicit scheme is usually possible if the computation time remains reasonable. However, in static or slow dynamic computations as duration of the study is large, many cycles are necessary to carry out the simulation.

To resolve static problems, an alternative to explicit method is the implicit time-integration scheme. In this method, a system of nonlinear equations is obtained and then resolved by Newton-Raphson method. It can be shown that the implicit scheme is always stable. That results in a large time step with the explicit method. However, as a global stiffness matrix should be assembled and inverted, the method is relatively high cost per loading step.

The primary difference between the explicit and implicit methods is that an explicit algorithm obtains the next value from known previous values. An implicit method assumes a solution to a problem and solves the equations simultaneously. As the global equilibrium equation is generally nonlinear, an iterative numerical resolution is generally used.

The implicit method might fail when:

- The material law is highly nonlinear. Complicated material behavior is easier to accommodate using an explicit method.
- The number of elements is too large.
- Explicit method does not require large matrix inversion, the I/O is less important and the memory required is also less.
- Matrices must be re-evaluated at each time step and for most of the iterations.

In such cases the CPU time of an explicit solution becomes competitive:

- The problem includes several contacts. Contact algorithms are very efficient in explicit programs.
- The static analysis is a pre-loading case before a fully dynamic behavior phase. In this case, the coupling of two phases is very common.
- Explicit approaches furnish an alternative to the previous cases.

As of RADIOSS V5 both of implicit and explicit methods are available to study the static behavior of systems. The choice a method depends on the nature of the problem and the engineer's feeling. The explicit approach is especially attractive for problems with highly nonlinear geometric and material behavior as all quantities may be treated as vectors, resulting in low storage requirements. The number of cycles to achieve convergence may be quite large, but global efficiency is generally observed. The implicit method is introduced to study efficiently static applications such as spring back in sheet metal forming or gravity loading or other initial state computations before / after dynamic simulations.

11.1 Static solution by explicit time-integration

Explicit algorithms are very useful for modeling a dynamic simulation. However, they cannot model a quasi-static or static simulation as easily. This is due the fact that in an explicit approach, first the nodal accelerations are found by resolving the equilibrium equation at time t_n. Other d.o.f's are then computed by explicit time integration. This procedure implies that the nodal acceleration must exist; however, some numerical methods may be employed for the simulation of a static process:

- **Slow dynamic computation**

The loading is applied at a rate sufficiently slow to minimize the dynamic effects. The final solution is obtained by smoothing the curves.

In case of elasto-plastic problems, one must minimize dynamic overshooting because of the irreversibility of the plastic flow.

- **Dynamic relaxation or Nodal Damping (/DYREL)**

This method was first introduced by Otter [19] and has been used in several hydrodynamic codes (e.g. HEMP [2]). A nodal damping *Cv* is added to the momentum equation:

$$M\frac{dv}{dt}+C\upsilon=f^{ext}-f^{\text{int}} \quad \text{EQ. 11.1.1.1}$$

The dashpot force is calculated by:

$$F_c=-CV \quad \text{EQ. 11.1.1.2}$$

The internal force is calculated by:

$$F_k=-KX=F_{kt-\Delta t/2}+KV\Delta t \quad \text{EQ. 11.1.1.3}$$

$$\gamma_0=\frac{F_k}{M} \quad \text{EQ. 11.1.1.4}$$

The total acceleration is given by:

$$\gamma_1=\gamma_0+\frac{F_c}{M}=\gamma_0-\frac{CV}{M} \quad \text{EQ. 11.1.1.5}$$

$$V_{t+\Delta t/2}=V_{t-\Delta t/2}+\gamma_1\Delta t \quad \text{EQ. 11.1.1.6}$$

$$V=V_{t-\Delta t/2}+\frac{1}{2}\gamma_1\Delta t \quad \text{EQ. 11.1.1.7}$$

$$V_{t+\Delta t/2}=V_{t-\Delta t/2}+\gamma_0\Delta t-\frac{CV}{M}\Delta t \quad \text{EQ. 11.1.1.8}$$

You have:

$$V_{t+\Delta t/2}=V_{t-\Delta t/2}+\gamma_0\Delta t-\frac{C}{M}\left(V_{t-\Delta t/2}+\frac{1}{2}\gamma_1\Delta t\right)\Delta t \quad \text{EQ. 11.1.1.9}$$

$$V_{t+\Delta t/2}=\left(1-\frac{C\Delta t}{M}\right)V_{t-\Delta t/2}+\left(\gamma_0-\frac{1}{2}\gamma_1\Delta t\frac{C}{M}\right)\Delta t \quad \text{EQ. 11.1.1.10}$$

$$V_{t+\Delta t/2}=\left(1-\frac{C\Delta t}{M}\right)V_{t-\Delta t/2}+\left(1-\Delta t\frac{C\gamma_1}{M\gamma_0}\right)\gamma_0\Delta t \quad \text{EQ. 11.1.1.11}$$

Approximation $\frac{\gamma_1}{\gamma_0}=1$ after the variable is changed, $\omega=C\frac{\Delta t}{2M}$, you obtain :

$$V_{t+\Delta t/2}=(1-2\omega)V_{t-\Delta t/2}+(1-\omega)\gamma_0\Delta t \quad \text{EQ. 11.1.1.12}$$

$$\omega=C\frac{\Delta t}{2M}\rightarrow C=\frac{2M\omega}{\Delta t} \quad \text{EQ. 11.1.1.13}$$

Which gives the expression of *C* as a proportional matrix to *M* with:

$$C = AM \quad \text{and} \quad A = \frac{2\omega}{\Delta t} \qquad \text{EQ. 11.1.1.14}$$

$$\text{or } A = \frac{2\beta}{\mathrm{T}} \qquad \text{EQ. 11.1.1.15}$$

$$\omega = \beta \frac{\Delta t}{\mathrm{T}} \qquad \text{EQ. 11.1.1.16}$$

Combining EQ. 11.1.1.14 and 15, and you obtain:

$$C = \frac{2\beta M}{\mathrm{T}} \qquad \text{EQ. 11.1.1.17}$$

Where β is the relaxation coefficient whose recommended value is 1. T is less than or equal to the highest period of the system. These are the input parameters used in /DYREL option.

The explicit time integration scheme is changed to compute the new velocities. The explicit time integration in Chapter 4 gives (EQ. 4.1.2.7):

$$V_{t+\Delta t/2} = V_{t-\Delta t/2} + \gamma_t \Delta t \qquad \text{EQ. 11.1.1.18}$$

which is now written as:

$$V_{t+\Delta t/2} = (1-2\omega) V_{t-\Delta t/2} + (1-\omega)\gamma_t \Delta t \qquad \text{EQ. 11.1.1.19}$$

where:

$$\omega = \beta \frac{\Delta t}{\mathrm{T}} \qquad \text{EQ. 11.1.1.20}$$

- **Energy discrete relaxation**

This empirical methodology consists in setting to zero the nodal velocities each time the Kinetic Energy reaches a maximum.

- **Rayleigh damping**

In this method a proportional damping matrix is defined as:

Table 1: $$[C] = \alpha[M] + \beta[K] \qquad \text{EQ. 11.1.1.21}$$

where α and β are the pre-defined constants. In modal analysis, the use of a proportional damping matrix allows to reduce the global equilibrium equation to n-uncoupled equations by using an orthogonal transformation.

If the global equilibrium equation is expressed as:

$$[M]\{\ddot{X}\} + [C]\{\dot{X}\} + [K]\{X\} = \{F_t\} \qquad \text{EQ. 11.1.1.22}$$

The transformed uncoupled system of equations can be written as:

$$[\phi]^T[M][\phi]\{\ddot{\xi}\}+[\phi]^T[C][\phi]\{\dot{\xi}\}+[\phi]^T[K][\phi]\{\xi\}=[\phi]^T\{F_t\}$$ EQ. 11.1.1.23

With $$[\phi]^T[C][\phi]=\begin{bmatrix} \alpha+\beta\omega_1^2 & 0 & . & . & 0 \\ 0 & \alpha+\beta\omega_2^2 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & . & . & . & \alpha+\beta\omega_n^2 \end{bmatrix}$$ EQ. 11.1.1.24

Each uncoupled equation is written as:

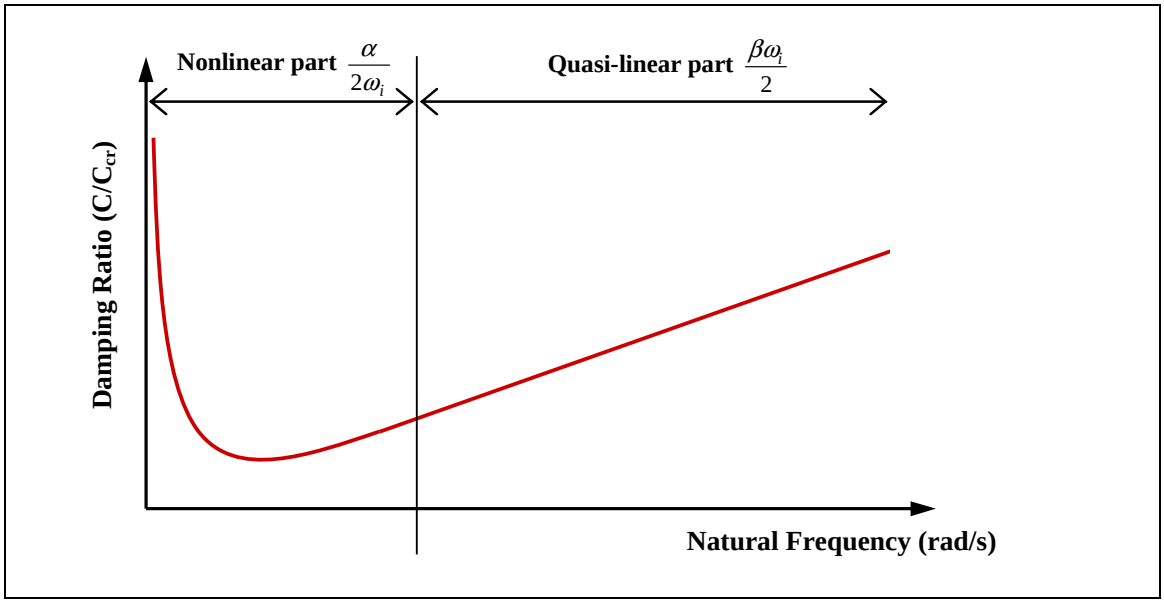
$$\ddot{\xi}_i+2\omega_i\zeta_i\dot{\xi}_i+\omega_i^2\xi_i=f_i^t$$ EQ. 11.1.1.25

With $$2\zeta_i\omega_i=\alpha+\beta\omega_i^2$$ EQ. 11.1.1.26

Where ω_i is the i[th] natural frequency of the system and ζ_i i[th] damping ratio.

This leads to a system of n equations with two unknown variables α and β. Regarding to the range of the dominant frequencies of system, two frequencies are chosen. Using the pair of the most significant frequencies, two equations with two unknown variables can be resolved to obtain values for α and β. For high frequencies the role of β is more significant. However, for lower frequencies α plays an important role (Fig. 11.1.1).

Fig. 11.1.1 Rayleigh damping variation for natural frequencies



The Rayleigh damping method applied to explicit time-integration method leads to the following equations:

$$M\gamma^t+Cv^t=F_{ext}^t-F_{\text{int}}^t$$ EQ. 11.1.1.27

With $C = \alpha M + \beta K$

$$F_{\text{int}}^{t} = F_{\text{int}}^{t-dt} + Kv^{t-\frac{dt}{2}}dt$$ EQ. 11.1.1.28

$$v^{t} = v^{t-\frac{dt}{2}} + \gamma^{t}\frac{dt}{2}$$ EQ. 11.1.1.29

$$M\gamma_0^{t} = F_{ext}^{t} - F_{\text{int}}^{t}$$ EQ. 11.1.1.30

Neglecting $F_{ext}^{t} - F_{ext}^{t-dt}$ and $v^{t} - v^{t-\frac{dt}{2}}$, in $\beta K v^{t}$ evaluation you have:

$$Kv^{t} = Kv^{t-\frac{dt}{2}} = \frac{F_{\text{int}}^{t} - F_{\text{int}}^{t-dt}}{dt} \approx M\frac{\left(\gamma_0^{t} - \gamma_0^{t-dt}\right)}{dt}$$ EQ. 11.1.1.31

And finally:

$$\gamma_0^{t} = M^{-1}\left(F_{ext}^{t} - F_{\text{int}}^{t}\right)$$ EQ. 11.1.1.32

$$\gamma^{t} = \frac{\gamma_0 - \alpha v^{t-\frac{dt}{2}} - \frac{\beta}{dt}\left(\gamma_0^{t} - \gamma_0^{t-dt}\right)}{1+\alpha\frac{dt}{2}}$$ EQ. 11.1.1.33

$$v^{t+\frac{dt}{2}} = v^{t-\frac{dt}{2}} + \gamma^{t}dt$$ EQ. 11.1.1.34

The three approaches available in RADIOSS are Dynamic Relaxation (/DYREL), Energy Discrete Relaxation (/KEREL) and Rayleigh Damping (/DAMP). Refer to *RADIOSS Example Manual* for application examples.

11.1.1 Acceleration convergence

For every method, an acceleration of the convergence to the static solution is desirable. The constant time step is one of the more usual methods. In fact, in quasi-static analysis, the duration of the study is proportional to the maximum period of the structure. The total number of computation cycles is then proportional to the ratio T/dt where T is largest period of the structure and *dt* the time step. The number of time steps necessary to reach the static solution is minimal if all the elements have the same time step. An initial given time step Δt_0 can be obtained by increasing or decreasing the density of each element. The constant nodal time step option ensures a homogenous time step over the structure. However in usual static problems the change is expected to be small, but one may think of increasing the density of the element which gives the critical time step in such a way that $\Delta t = \Delta t_0$.

11.2 Static solution by implicit time-integration

The static behavior of many structures can be characterized by a load-deflection or force-displacement response. If the response plot is nonlinear, the structure behavior is nonlinear. From computational point of view the resolution of a nonlinear problem is much more complex with respect to the linear case. However, the use of relatively recent resolution methods based on sparse iterative techniques allows saving substantially in memory.

11.2.1 Linear static solver

A linear structure is a mathematical model characterized by a linear fundamental equilibrium path for all possible choices of load and deflection variables. This implies that:

- The response to different load systems can be obtained by superposition,
- Removing all loads returns the structure to the reference position.

The requirements for such a model to be applicable are:

- Perfect linear elasticity for any deformation,
- Infinitesimal deformation,
- Infinite strength.

Despite of obvious physically unrealistic limitations, the linear model can be a good approximation of portions of nonlinear response. As the computational methods for linear problems are efficient and low cost, RADIOSS linear solvers can be used to find equilibrium of quasi-linear systems. The Preconditioned Conjugate Gradient method is the iterative linear solver available in RADIOSS. The algorithm enables saving a lot of memory for usual application of RADIOSS as a sparse storage method is used. This means that only the non-zero terms of the global stiffness matrix are saved. In addition, the symmetry property of both stiffness and preconditioning matrices is worthwhile to save memory.

The performance of conjugate gradient method depends highly to the preconditioning method. Several options are available in RADIOSS using the card /IMPL/SOLV/1. The simplest method is a so-called Jacobi method in which only the diagonal terms are taken into account. This choice allows saving considerable memory space; however, the performance may be poor. The incomplete Choleski is one of the best known effective preconditioning methods. However, it can result in negative pivots in some special cases even if the stiffness matrix is definite positive. This results a low convergence of PCG algorithm. The problem can be resolved by using a stabilization method as described in [103]. Finally, the Factored Approximate Inverse method may be the best choice which is used by default in RADIOSS.

11.2.2 Nonlinear static solver

As explained in the beginning of this chapter, a nonlinear behavior is characterized by a nonlinear load-deflection diagram called *path*. The tangent to an equilibrium path may be formally viewed as the limit of the ratio force increment on displacement increment. This is the definition of a stiffness or more precisely the tangent stiffness related to a given equilibrium state. The reciprocal ratio is called flexibility. The sign of the tangent stiffness is closely associated with the stability of an equilibrium state. A negative stiffness is necessary associated with unstable equilibrium. A positive stiffness is necessary but not sufficient for stability.

The problem of nonlinear analysis can be viewed as that of minimising the total potential energy Π which is a function of the total displacement **X**. A truncated Taylor series then leads to:

$$\Pi_n(\mathrm{X}+\delta\mathrm{X}) = \Pi_0(\mathrm{X}) + \frac{\partial\Pi}{\partial\mathrm{X}}\delta\mathrm{X} + \frac{1}{2}\delta\mathrm{X}^{\mathrm{T}}\frac{\partial^2\Pi}{\partial\mathrm{X}^2}\delta\mathrm{X} + \ldots \qquad \text{EQ. 11.2.2.1}$$

where the subscripts *n* and 0 denote respectively final and initial configurations. The term $\frac{\partial\Pi}{\partial\mathrm{X}}$ can be identified as the out-of-balance forces or gradient **F**, of the total potential energy which is the difference between the internal force vector $\mathbf{F}_{\mathrm{int}}$ and the external force vector $\mathbf{F}_{\mathrm{ext}}$. The term $\frac{\partial^2\Pi}{\partial\mathrm{X}^2}$ describes the tangent stiffness matrix K_T. The principle of minimum energy and the equilibrium of stable state give:

$$\delta\Pi = \Pi_n(\mathrm{X}+\delta\mathrm{X}) - \Pi_0(\mathrm{X}) = 0 \qquad \text{EQ. 11.2.2.2}$$

which is implied in EQ. 11.2.2.1:

$$K_T \delta X = F_{ext}(X) - F_{int}(X) \quad \text{EQ. 11.2.2.3}$$

The tangent matrix K_T should be defined positive at the equilibrium point for stable case:

$$\delta X^T K_T \delta X > 0 \quad \text{EQ. 11.2.2.4}$$

EQ. 11.2.2.3 can handle the solution of the nonlinear problem when an incremental method is used. The solution methods are generally based on continuous incremental and corrective phases. The most important class of corrective methods concerns the Newton-Raphson method and its numerous variants as modified, modified-delayed, damped, quasi and so forth. All of these Newton-like methods require access to the past solution. In the following section the conventional and modified Newton methods under general increment control are studied.

11.2.2.1 Newton and modified Newton methods

As you will often prefer to trace the complete load/deflection response or in other words, the equilibrium path, it is useful to combine the incremental and iterative solution procedures. You can recall that the purpose is to solve EQ.11.2.2.3 which can be written in residual form:

$$R(X,\lambda) = F_{ext}(X,\lambda) - F_{int}(X) = 0 \quad \text{EQ. 11.2.2.5}$$

with $F_{ext}(X,\lambda) = F_C + \lambda F_{ext}(X)$. This equation represents a system of n algebraic nonlinear equations depending on only one loading parameter λ. If the loading depends to only one loading variable independent to the state of deflection, you have:

$$F_{ext}(X,\lambda) = \lambda F_{ext}(X) \quad \text{EQ. 11.2.2.6}$$

Several techniques are available to resolve EQ. 11.2.2.5. In some situations, the parameter λ is fixed, and the equations are resolved to determine n components of **X** in order to verify EQ. 11.2.2.5. In this case, the technique is called *load control* method. Another technique called *displacement control* consists in fixing a component of **X** and searching for λ and 'n-1' other components of displacement vector **X**. A generalization of displacement control technique will enable to imply several components of displacement vector by using an Euclidian norm. The method is called *arc-length control* and intended to enable solution algorithms to pass limit points (i.e. maximum and minimum loads). The techniques making possible to obtain the load-deflection curve by finding point by point the solution are called *piloting techniques.*

When the piloting technique is chosen for a given step, the associated solution is obtained by an iterative resolution of so-called Newton-Raphson methods. At iteration i, the residual vector $\mathbf{R^i}$ is:

$$R^i = F_{ext}(X^i, \lambda^i), -F_{int}(X^i) \quad \text{EQ. 11.2.2.7}$$

A correction ΔX and $\Delta\lambda$ can be considered with:

$$R^{i+1} = R^i + \left[\frac{\partial R}{\partial X}\right]^i \Delta X + \left[\frac{\partial R}{\partial \lambda}\right]^i \Delta\lambda \quad \text{EQ. 11.2.2.8}$$

Combining EQ.11.2.2.8 with EQ. 11.2.2.7, youobtain:

$$K_T^i \Delta X - F_{ext}^i \Delta\lambda = R^i \quad \text{EQ. 11.2.2.9}$$

as $R^{i+1} = 0$ and:

$$X^{i+1} = X^i + \Delta X$$

$$\lambda^{i+1} = \lambda^{i} + \Delta\lambda$$ EQ. 11.2.2.10

The tangent matrix $K_T^{\ i}$ is obtained by assembling the elementary matrices $k_T^{\ i}$. It corresponds to:

$$K_T^{\ i} = \frac{\partial F_{int}}{\partial X} - \frac{\partial F_{ext}}{\partial X}$$ EQ. 11.2.2.11

Using load control technique, the standard Newton-Raphson method resolves EQ. 11.2.2.9 to EQ. 11.2.2.11 by applying a known load increment $\Delta\lambda$ as illustrated in Figure 11.2.1. The tangent matrix is updated and triangulized at each iteration. This insures a quadratic convergence to exact solution.

Figure 11.2.1 Standard Newton-Raphson resolution in the case of load control technique

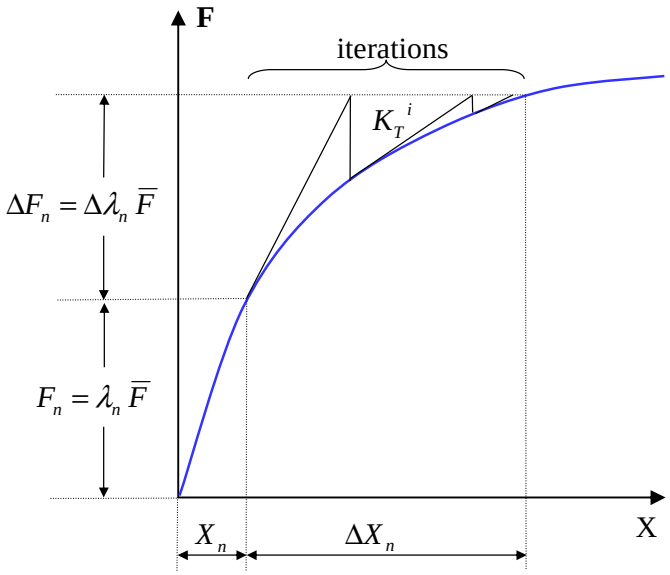
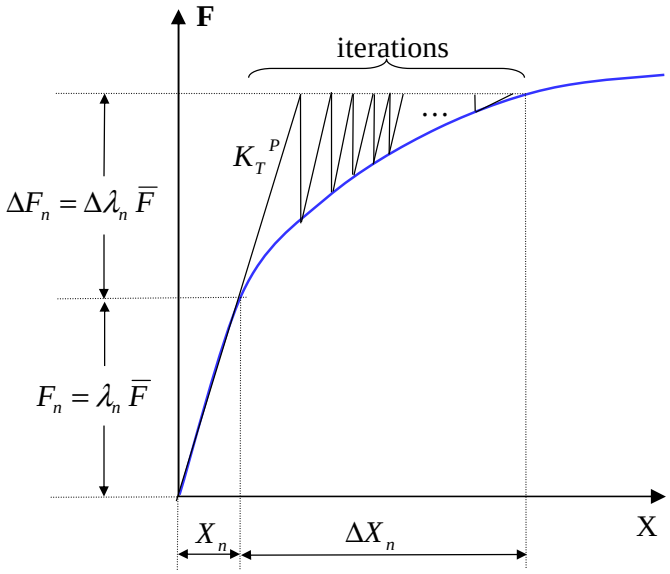


Figure 11.2.2 Modified Newton-Raphson resolution in the case of load control technique



However, it is possible to save computation time which depends on the size of the problem and on the degree of the nonlinearity of the problem. The method is called *modified Newton-Raphson* which is based on the conservation of the tangent matrix for all iterations (Figure 11.2.2). This method can also be combined with the acceleration techniques as *line-search* explained in the following section.

The convergence criteria may be based on Euclidian norm of residual forces, residual displacements or energy where an allowable tolerance is defined.

11.2.2.2 Line search method to optimise the resolution

The Newton-Raphson resolution of EQ. 11.2.2.9 implies updating the variables at each iteration with EQ. 11.2.2.10. The new estimation of X^{i+1} does not satisfy EQ. 11.2.2.9 only if $R^{i+1}=0$. In order to reduce the number of iterations the line-search method is used. The line-search technique is an important feature of most numerical techniques used in optimisation problems. Detailed discussions are given in [104]. The method consists in introducing a parameter α such as:

$$X^{i+1} = X^{i} + \alpha \Delta X \qquad \text{EQ. 11.2.2.12}$$

where α is obtained to minimize the total potential energy or to satisfy the principle of virtual works. The techniques to determine α use often a Raleigh-Ritz procedure with only one unknown parameter.

The principle of virtual work can be written in the general form:

$$W(X,\delta X) = \delta X \bullet R(X) = 0 \quad \text{For all kinematical acceptable } \delta X \qquad \text{EQ. 11.2.2.13}$$

Considering EQ. 11.2.2.12, write:

$$\delta X = \alpha \Delta X \qquad \text{EQ. 11.2.2.14}$$

and:

$$\delta W = \delta\alpha \Delta X \bullet R\left(X^{i} + \alpha \Delta X\right) = 0 \quad \text{for all } \alpha \qquad \text{EQ. 11.2.2.15}$$

Then, α is determined from:

$$\Delta X \bullet R\left(X^{i} + \alpha \Delta X\right) = 0 \qquad \text{EQ. 11.2.2.16}$$

which leads to a three-order polynomial equation in α for elastic materials:

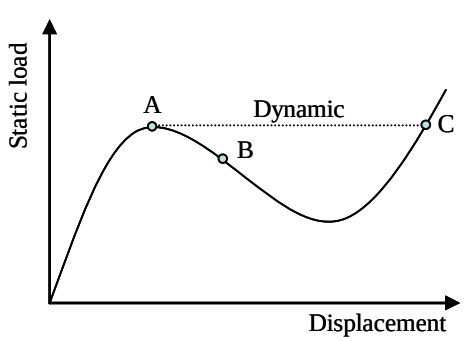
$$C_1 + C_2\alpha + C_3\alpha^2 + C_4\alpha^3 = 0 \qquad \text{EQ. 11.2.2.17}$$

The coefficients C1, C2, C3 and C4 can be expressed in terms of displacements X^i and the increment of displacements ΔX.

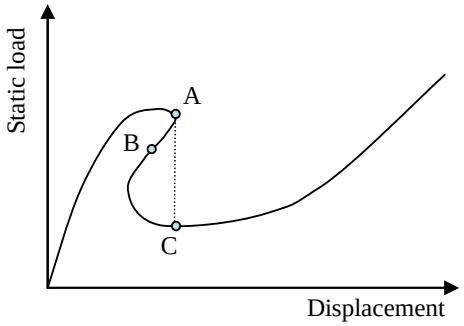
11.2.2.3 Arc length method

To obtain the load-deflection behavior of a structure, the load or the displacement of a given point of the structure must be parameterized. Up to now, you have parameterized the load by the time *t*. However, a single parameter is not always sufficient to control in an optimum way the time step. On the other hand, it is not possible to pass limit points with ‘snap-through’ and ‘snap-back’ when using load-controlled or displacement-controlled techniques. This is due to the fact that the increase in load or in a given displacement component may result a dynamic response losing a part of load-deflection curve as shown in Figures 11.2.3 (a) and (b).

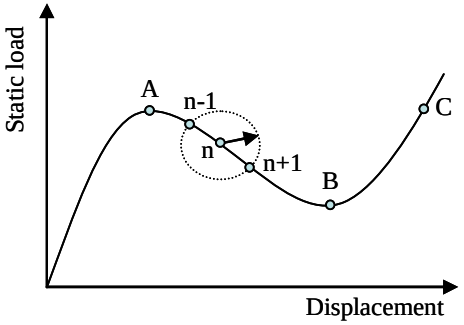
Figure 11.2.3 Various load-deflection curves and step-by-step solution by arc-length method



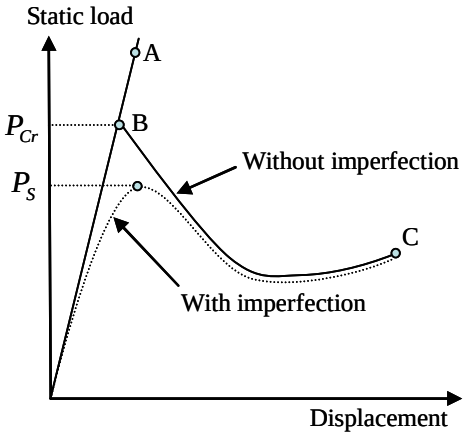
(a) Snap-through



(b) Snap-backs



(c) Arc-length method: intersection of the equilibrium branch with the circle about the last solution



(d) Buckling with or without imperfections

The tracing of equilibrium branches are quite difficult. In arc-length method, instead of incrementing the load parameter, a measure of the arc length in the displacement-load parameter space is incremented. This is accomplished by adding a controlling parameter to the equilibrium equations.

The arc-length method was originally introduced by Riks [116] and Wempner [117]. Considering a function *f* implying several components of the displacement vector *X*, the arc-length method consists in determining in each step the Euclidian norm of the increase in *X*:

$$f = \left\langle X^{i+1} - X^{n} \right\rangle \left\{ X^{i+1} - X^{n} \right\} - (\Delta S_n)^2 = 0 \qquad \text{EQ. 11.2.2.18}$$

This leads to:

$$\left\langle X^{i+1} - X^{n} \right\rangle \left\{ X^{i+1} - X^{n} \right\} = (\Delta S_n)^2 \qquad \text{EQ. 11.2.2.19}$$

And $$a(\Delta\lambda)^2 + b(\Delta\lambda) + c = 0 \qquad \text{EQ. 11.2.2.20}$$

With:

$$a = \langle \Delta X_F \rangle \{\Delta X_F\} ; \qquad \{\Delta X_F\} = \left[K_T^n\right]^{-1} \{F_{ext}\}$$

$$b = 2\langle \Delta X_F \rangle \{Y\} ; \qquad \{Y\} = \{\Delta X_F\} + \left\{X^i - X^n\right\}$$

$$c = \langle Y \rangle \{Y\} - (\Delta S_n)^2$$

In each of the Newton-Raphson iterations, EQ. 11.2.2.20 must be resolved to select a real root. If there is no root, ΔS_n should be reduced. The most closed root to the last solution is retained in the case of two real roots.

Figure 11.2.3(c) illustrates the intersection of the equilibrium branch with the circle about the last solution.